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BOUNDARY VALUE PROBLEMS OF THE CALCULUS OF VARIATIONS†

BY W. T. REID

1. *Introduction.* The term boundary value problem is applied to the question of determining whether a given system of differential equations (in general, a given system of functional equations) has one or more solutions satisfying certain prescribed end or boundary conditions; and, if so, the determination of the character of these solutions and how their character changes when the differential equations or boundary conditions change. This address is restricted to the discussion of a class of linear boundary problems which are intimately associated with the calculus of variations. These boundary value problems have been used in establishing sufficient conditions, especially for the more complicated problems of the calculus of variations. On the other hand, the principles and theorems of the calculus of variations have been of extreme significance in the advancement of the theory of such boundary value problems. In fact, the calculus of variations has served to unify a certain class of boundary problems much larger than that seemingly represented by the problem that we shall first formulate. In view of the rather extensive interest and study of these problems within recent years, it seems proper at this time to discuss the present status of such boundary problems, to compare the various methods that have been used in their treatment, and to indicate various questions concerning them that are as yet unsolved.

Historically, the study of boundary problems associated with a second order linear differential equation dates from the time of Euler and D'Alembert. The first somewhat general theory of such problems, however, is that given by Sturm [1]‡ in his fundamental memoir of 1836. One of the most important questions for the more complicated boundary problems that we shall here consider is that of generalizing the Sturm oscillation and

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‡ Numerals in square brackets refer to the bibliography at the end of the paper.

comparison theorems for a single second order equation. Consequently, I shall describe them briefly. Perhaps the most fundamental result of Sturm's paper is the following comparison theorem:

Suppose u_1 and u_2 are solutions of the equations

$$(1)\qquad \begin{aligned}(K_1u_1')' - G_1u_1 &= 0,\\ (K_2u_2')' - G_2u_2 &= 0,\end{aligned}$$

where $K_1 \geqq K_2 > 0$, $G_1 \geqq G_2$ and both equality signs do not hold throughout x_1x_2, while $u_1(x_1) = 0 = u_1(x_2)$. Then u_2 vanishes at least once on $x_1 < x < x_2$.

The general oscillation theorem of Sturm's paper is concerned with a differential equation

$$(2)\qquad (K(x, \lambda)u')' - G(x, \lambda)u = 0$$

with two-point boundary conditions of the form

$$(3)\qquad \begin{aligned}\alpha_1(\lambda)u(x_1) - \beta_1(\lambda)K(x_1, \lambda)u'(x_1) &= 0,\\ \alpha_2(\lambda)u(x_2) + \beta_2(\lambda)K(x_2, \lambda)u'(x_2) &= 0.\end{aligned}$$

Under certain general conditions, which include in particular the assumption that $K(x, \lambda)$, $G(x, \lambda)$, $\alpha_1(\lambda)/\beta_1(\lambda)$, and $\alpha_2(\lambda)/\beta_2(\lambda)$ are monotone decreasing functions of λ, it is established that there exists an infinity of characteristic values $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$, and that the characteristic solution u_n corresponding to λ_n has exactly $n-1$ zeros on $x_1 < x < x_2$. It is to be noted that (2) is the Euler equation and (3) the transversality conditions associated with the functional

$$(4)\qquad \begin{aligned}&[\alpha_1(\lambda)/\beta_1(\lambda)]u^2(x_1) + [\alpha_2(\lambda)/\beta_2(\lambda)]u^2(x_2)\\ &\qquad + \int_{x_1}^{x_2} [K(x, \lambda)u'^2 + G(x, \lambda)u^2]dx.\end{aligned}$$

By way of introduction, consider the problem of minimizing the integral

$$(5)\qquad \mathfrak{I} = \int_{x_1}^{x_2} f(x, y, y')dx$$

in the class of arcs y of the form $y = y(x)$ on $x_1 \leqq x \leqq x_2$ and joining the two points $P_1: (x_1, y_1)$ and $P_2: (x_2, y_2)$. Along a minimizing arc E the second variation

$$(6) \qquad \mathfrak{J}_2 = \int_{x_1}^{x_2} (f_{rr}\eta'^2 + 2f_{ry}\eta'\eta + f_{yy}\eta^2)dx$$

must be non-negative in the class of arcs $\eta = \eta(x)$, $x_1 \leqq x \leqq x_2$, such that $\eta(x_1) = 0 = \eta(x_2)$. Moreover, for the given arc E to be a minimizing arc it must be true that $f_{rr} \geqq 0$ along E. If we suppose that E is an extremal which is non-singular in the sense that along it we have $f_{rr} \neq 0$, then the usual Jacobi necessary condition states that there exists no non-identically vanishing solution $\eta(x)$ of the Jacobi equation

$$(7) \qquad J(\eta) \equiv (f_{rr}\eta' + f_{ry}\eta)' - (f_{ry}\eta' + f_{yy}\eta) = 0$$

such that $\eta(x_1) = 0 = \eta(x_3)$, and $x_1 < x_3 < x_2$. A point x_3 for which such a solution of (7) exists is said to determine a point $P_3: [x_3, y(x_3)]$ on E conjugate to P_1. If the strengthened Jacobi condition is satisfied, that is, if there is no value x_3 on $x_1 < x_3 \leqq x_2$ determining a point conjugate to P_1, the second variation $\mathfrak{J}_2[\eta]$ may be shown to be positive for all non-identically vanishing arcs.

Now consider the boundary problem

$$(8) \qquad J(\eta) + \lambda\eta = 0, \qquad \eta(x_1) = 0 = \eta(x_2)$$

involving the characteristic parameter λ. A value λ is called a characteristic value of (8) if there exists a corresponding non-identically vanishing solution η of (8). One may show in a simple manner that all the characteristic values of (8) are real. If λ is a characteristic value of (8) and η a corresponding solution, it follows by integration by parts that

$$\mathfrak{J}_2[\eta] = \lambda \int_{x_1}^{x_2} \eta^2 dx.$$

Hence in order that $\mathfrak{J}_2 \geqq 0$ it is necessary that there exist no negative characteristic values of this system. Clearly x_2 determines P_2 conjugate to P_1 on E if and only if $\lambda = 0$ is a characteristic value of (8).

It is a consequence of Sturm's oscillation and comparison theorems that the Jacobi condition is equivalent to the condition that there exist no negative characteristic values of (8). Furthermore, if one utilizes expansion theorems for arbitrary functions $\eta(x)$ satisfying the end-conditions in terms of the char-

acteristic solutions of (8), it may be shown that the smallest characteristic value of this system is actually the minimum of $\mathfrak{I}_2[\eta]$ in the class of η's satisfying the end conditions and normed so that $\int_{x_1}^{x_2}\eta^2 dx = 1$. Now by the introduction of the Green's function the system (8) is seen to be equivalent to an integral equation with real symmetric kernel, and such an expansion theorem follows from the Hilbert-Schmidt theory of integral equations. Indeed, it is a ready consequence of the expansion theorem that each characteristic value of (8) is the minimum value of $\mathfrak{I}_2[\eta]$ in a suitable class of arcs η.

It is to be emphasized that the choice of the associated boundary value problem is somewhat arbitrary. In the above case one might equally well have considered the problem

$$J(\eta) + \lambda k(x)\eta = 0, \qquad \eta(x_1) = 0 = \eta(x_2) \tag{9}$$

where $k(x)$ is a positive function on x_1x_2. It is also to be noted that for an admissible arc η the second variation reduces on integration by parts to

$$\mathfrak{I}_2[\eta] = \int_{x_1}^{x_2} (f_{y'y'}(x)\eta'^2 + g\eta^2)dx, \qquad (g = f_{yy} - df_{yy'}/dx).$$

If one considers the boundary problem

$$(f_{y'y'}\eta')' - \lambda g\eta = 0, \qquad \eta(x_1) = 0 = \eta(x_2),$$

it may be shown that $\mathfrak{I}_2 \geqq 0$ (>0) if and only if the smallest positive characteristic value is $\geqq 1$ (>1). This form of the associated boundary value problem has been used by Lichtenstein for both simple and double integral problems of the calculus of variations.

For the more complicated problems of the calculus of variations one may associate with the second variation a linear boundary problem in a manner analogous to that indicated above for the plane problem. Such a formulation for the general problem of Mayer was first explicitly given by Cope [1]. Immediately there arises the question of the existence of characteristic values for this problem, and the relations between such a boundary problem and the positiveness of the second variation. Can one generalize the results of the Sturmian theory to this more general boundary problem and thus prove the existence of characteristic values and properties of correspond-

ing solutions? Actually, this order of proof has been carried through only recently. For boundary value problems associated with calculus of variations problems of varying complexity the existence of characteristic values has been established by various methods. Among the methods which have been used are the following, and various combinations of them: (1) the use of comparison and approximation theorems, (2) asymptotic expressions, (3) integral equations, (4) approximating algebraic difference equations, (5) the theory of linear algebraic equations in an infinite number of variables, (6) general theorems on differential equations, (7) the minimum principle of the calculus of variations.

From the standpoint of comparison and oscillation theorems for the more general boundary problems, it is to be remarked that either these theorems may first be proved and the existence of characteristic values deduced from them, or the existence of characteristic values may be first proved and these comparison and oscillation theorems then proved as a consequence of the extremizing properties of the characteristic values and solutions.

2. *Formulation of a General Boundary Problem.* Bliss [6] has emphasized the significant position of the so-called problem of Bolza with respect to the other problems of the calculus of variations. For brevity, the general problem of Bolza will not be formulated here. We shall, however, give the form of the second variation of such a problem, and discuss associated boundary value problems. The symbols $\eta \equiv [\eta_i]$, $\eta' \equiv [\eta_i']$ will denote the functions $[\eta_1(x), \cdots, \eta_n(x)]$ and the set of their derivatives, respectively. The second variation for a problem of Bolza may then be written

$$\mathfrak{I}_2[\eta] \equiv 2Q[\eta(x_1), \eta(x_2)] + \int_{x_1}^{x_2} 2\omega(x, \eta, \eta')dx, \tag{10}$$

where ω and Q are quadratic forms in the $2n$ variables η_i, η_i' and $\eta_i(x_1)$, $\eta_i(x_2)$, respectively. The associated equations of variation of the auxiliary differential and end-conditions of the problem of Bolza are of the form

$$\Phi_\alpha(x, \eta, \eta') \equiv \Phi_{\alpha\tau_i}(x)\eta_i' + \Phi_{\alpha\eta_i}(x)\eta_i = 0, \qquad (\alpha = 1, \cdots, m < n), \tag{11}$$

$$(12)\qquad \Psi_\gamma[\eta(x_1), \eta(x_2)] \equiv \Psi_{\gamma;i1}\eta_i(x_1) + \Psi_{\gamma;i2}\eta_i(x_2) = 0, \qquad (\gamma = 1, \cdots, p \leqq 2n).$$

It is to be remarked that in order to write the second variation of a problem of Bolza in the form (10) it is in general necessary to include in the set η not only the variations of the dependent functions in the original problem of Bolza, but also two new functions representing the variations of the end-values; these latter two functions are further restricted by including in the set (11) two additional differential equations which require them to be constant on x_1x_2 (see, for example, Reid [2]).

Partial derivatives of $\omega(x, \eta, \pi)$, $\Phi_\alpha(x, \eta, \pi)$ with respect to the variables η_i, π_i will be denoted by writing these variables as subscripts; derivatives of Q and Ψ_γ with respect to the arguments $\eta_i(x_1)$, $\eta_i(x_2)$ will be indicated by Q_{i1}, $\Psi_{\gamma;i1}$, Q_{i2}, $\Psi_{\gamma;i2}$, respectively. The summation convention of tensor analysis is used throughout.

The conditions usually required of an arc for the problem of Bolza in order to obtain sufficiency theorems insure for the second variation the following properties.

(H_1) The coefficients of the quadratic form $\omega(x, \eta, \pi)$ and the linear expressions $\Phi_\alpha(x, \eta, \pi)$ are real single-valued functions of class C^1, and the matrix $\|\Phi_{\alpha\pi_j}(x)\|$ has rank m on x_1x_2. The coefficients of Q and Ψ_γ are real constants, and the matrix $\|\Psi_{\gamma;i1}\Psi_{\gamma;i2}\|$ is of rank p.

(H_2) The quadratic form $\omega_{\pi_i\pi_j}(x)u_iu_j$ is positive for all values $(u_i) \neq (0_i)$ satisfying $\Phi_{\alpha\pi_j}(x)u_j = 0$, $(\alpha = 1, \cdots, m)$.

Condition (H_2) implies, in particular, that the matrix

$$(13)\qquad \begin{vmatrix} \omega_{\pi_i\pi_j} & \Phi_{\alpha\pi_i} \\ \Phi_{\beta\pi_j} & 0_{\beta\alpha} \end{vmatrix}, \qquad (i, j = 1, \cdots, n; \beta, \alpha = 1, \cdots, m),$$

is non-singular on x_1x_2.

An arc $\eta \equiv [\eta_i(x)]$ will be called *differentially admissible* if the functions $\eta_i(x)$ are of class D^1 and satisfy the equations (11) on x_1x_2. An arc whose end values satisfy equations (12) will be said to be *terminally admissible*. Finally, an arc which is both differentially and terminally admissible will, for brevity, be termed *admissible*.

(H_3) There exist p differentially admissible arcs $\eta_i = \eta_{i\nu}(x)$,

($\nu = 1, \cdots, p$), such that the determinant $|\Psi_\gamma[\eta_\nu(x_1), \eta_\nu(x_2)]|$ is different from zero.

Hypothesis (H_3) is a condition of normality with respect to the differential equations (11) and end conditions (12).

The accessory (Jacobi) system along a given non-singular extremal for the original problem of Bolza is defined as the Euler-Lagrange equations and transversality conditions for the corresponding second variation (10) and the auxiliary differential equations (11). If we set $\Omega(x, \eta, \pi, \mu) = \omega(x, \eta, \pi) + \mu_\alpha \Phi_\alpha(x, \eta, \pi)$, these differential equations are

$$J_i(\eta, \mu) \equiv d\Omega_{\pi_i}/dx - \Omega_{\eta_i} = 0, \qquad \Phi_\alpha = 0, \tag{14}$$
$$(i = 1, \cdots, n; \alpha = 1, \cdots, m);$$

the corresponding transversality conditions involve constants d_γ satisfying with the end values of η_i, μ_α the relations

$$Q_{is}[\eta] + d_\gamma \Psi_{\gamma i s} + (-1)^s \Omega_{\pi_i}(x_s) = 0, \qquad \Psi_\gamma = 0, \tag{15}$$
$$(s = 1, 2; \gamma = 1, \cdots, p).$$

The associated accessory boundary problem consists of the equations

$$J_i(\eta, \mu) + \lambda \eta_i = 0, \qquad \Phi_\alpha = 0 \tag{16}$$

involving the characteristic parameter λ, together with the end conditions (15). A value λ will be said to be a characteristic value of this problem if there exist corresponding solutions η_i, μ_α of (16) not all identically zero, and whose end values at x_1 and x_2 satisfy with suitable constants d_γ the end-conditions (15). The above hypothesis (H_3) insures the condition $\eta_i \eta_i \not\equiv 0$ on $x_1 x_2$ for an arbitrary solution η_i, μ_α of this system. By a method similar to that indicated in the introduction for the simple problem of the calculus of variations, one may show that in order for $\mathfrak{I}_2[\eta]$ to be non-negative in the class of admissible arcs η it is necessary that there exist no negative characteristic value of (16), (15). In view of the minimizing properties of the characteristic values of this system to be discussed later, one may also state that the non-existence of negative characteristic values of the accessory problem is equivalent to the non-negativeness of $\mathfrak{I}_2[\eta]$ in the class of admissible arcs η.

Since the matrix (13) is assumed to be non-singular, canonical variables ζ_i may be introduced by means of the equations

$$\zeta_i = \Omega_{\pi_i}(x, \eta, \pi, \mu), \qquad 0 = \Phi_\alpha(x, \eta, \pi),$$
$$(i = 1, \cdots, n; \alpha = 1, \cdots, m),$$

and system (16) may be written as

$$(17) \quad \eta_i' = A_{ij}(x)\eta_j + B_{ij}(x)\zeta_j, \ \zeta_i' = C_{ij}(x)\eta_j - A_{ji}(x)\zeta_j - \lambda\eta_i.$$

The elements A_{ij}, B_{ij}, C_{ij} in (17) are continuous, the matrices $\|B_{ij}\|$, $\|C_{ij}\|$ are symmetric, and $\|B_{ij}\|$ is of rank $n-m$ on x_1x_2. Correspondingly, the end conditions (15) reduce to

$$(18) \quad s_\sigma[\eta, \zeta] \equiv a_{\sigma i}\eta_i(x_1) - b_{\sigma i}\zeta_i(x_1) + c_{\sigma i}\eta_i(x_2) + d_{\sigma i}\zeta_i(x_2),$$
$$(\sigma = 1, \cdots, 2n),$$

where the coefficients of s_σ are such that the matrix $\|a_{\sigma i}b_{\tau i} + c_{\sigma i}d_{\tau i}\|$, $(\sigma, \tau = 1, \cdots, 2n)$, is symmetric. Moreover, for an arbitrary system of the form (17), (18) whose coefficients satisfy the above conditions there exist quadratic forms $Q[\eta(x_1), \eta(x_2)]$, $\omega(x, \eta, \eta')$ with associated linear equations $\Phi_\alpha(x, \eta, \eta') = 0$, $(\alpha = 1, \cdots, m)$, and end conditions $\Psi_\gamma = 0$ such that the given system is derived from this associated system (10), (11), (12) in the manner described above (see Hu [1]).

It is to be noted that the equations (16), (15) are the Euler-Lagrange equations and transversality conditions for the functional

$$(19) \qquad \mathfrak{I}_2[\eta] - \lambda \int_{x_1}^{x_2} \eta_i\eta_i dx$$

subject to the differentiai equations (11) and end-conditions (12). The more general boundary problem to be considered consists of these equations for a functional

$$(20) \quad \mathfrak{I}_2[\eta:\lambda] = 2Q[\eta(x_1), \eta(x_2):\lambda] + \int_x^{x_2} 2\omega(x, \eta, \eta':\lambda)dx$$

subject to (11) and (12). The canonical form of this problem is then

$$(21) \quad \eta_i' = A_{ij}(x:\lambda)\eta_j + B_{ij}(x:\lambda)\zeta_j, \ \zeta_i' = C_{ij}(x:\lambda)\eta_j - A_{ji}(x:\lambda)\zeta_j,$$
$$s_\sigma[\eta, \zeta:\lambda] = 0.$$

For the still more general problem in which the coefficients of (11) and (12) also depend on λ, at present no important re-

sults concerning the existence of characteristic values and associated comparison and oscillation theorems have been obtained. In fact, one may show by simple examples that these conditions as usually phrased are in general not satisfied by such problems.

Of particular significance is the special problem of the form (21) in which this system reduces to

$$(22)\quad \eta_i' = A_{ij}(x)\eta_j + B_{ij}(x)\zeta_j,\ \zeta_i' = C_{ij}(x)\eta_j - A_{ji}(x)\zeta_j - \lambda K_{ij}\eta_j,$$
$$s_\alpha[\eta, \zeta] = 0,$$

that is, the case in which Q is independent of λ and $2\omega(x, \eta, \eta' : \lambda) \equiv 2\omega(x, \eta, \eta') - \lambda\eta_i K_{ij}(x)\eta_i$.

The above assumption (H_2) is equivalent to the condition that there is no characteristic solution η_i, ζ_i for which $\eta_i \equiv 0$ on x_1x_2 (see, for example, Bliss [4], p. 48). As is customary (Reid [4], p. 575), we shall say that the order of anormality on x_1x_2 of the differential equations of the above boundary problem is equal to r if on this interval they possess exactly r linearly independent solutions of the form $\eta_i \equiv 0$, $\zeta_i = v_{ih}(x)$, $(h = 1, \cdots, r)$. Since the expressions Φ_α do not depend on λ, the value of r is independent of λ. It follows readily that $r \leqq m$; moreover, if η is an arbitrary differentially admissible arc and $x'x''$ are any two points of x_1x_2, then

$$v_{ih}(x)\eta_i(x)\Big|_{x'}^{x''} = 0, \qquad (h = 1, \cdots, r).$$

As a consequence of these relations, hypothesis (H_3) is equivalent to the assumption that the matrix

$$(23)\qquad \left\| \begin{matrix} \Phi_{\gamma i j 1} & \Psi_{\gamma i j 2} \\ -v_{jh}(x_1) & v_{jh}(x_2) \end{matrix} \right\|,$$
$$(j = 1, \cdots, n;\ h = 1, \cdots, r;\ \gamma = 1, \cdots, p),$$

has rank $p+r$. This implies, in particular, that $p \leqq 2n-r$.

Now the class of admissible arcs for a problem is unchanged when the conditions $\Psi_\gamma = 0$ are replaced by conditions of the form

$$\Psi_\gamma + e_{\gamma h}[-v_{jh}(x_1)\eta_j(x_1) + v_{jh}(x_2)\eta_j(x_2)] = 0,\quad (\gamma = 1, \cdots, p),$$

where $e_{\gamma h}$, $(\gamma = 1, \cdots, p;\ h = 1, \cdots, r)$, are arbitrary constants. If η_i, ζ_i is a characteristic solution of the original problem, there

are constants e_h such that $\eta_i, \zeta_i + v_{ih}e_h$ is a characteristic solution of the modified problem. Consequently, we do not distinguish between two problems which satisfy the above hypotheses and differ only in the end-conditions $\Psi_\gamma^1 = 0$ and $\Psi_\gamma^2 = 0$, $(\gamma = 1, \cdots, p)$, respectively, and these conditions are such that the matrix

$$\begin{vmatrix} \overset{1}{\Psi}_{\gamma;j1} & \overset{1}{\Psi}_{\gamma;j2} \\ \overset{2}{\Psi}_{\gamma;j1} & \overset{2}{\Psi}_{\gamma;j2} \\ -v_{jh}(x_1) & v_{jh}(x_2) \end{vmatrix}, \tag{24}$$

$$(j = 1, \cdots, n; \gamma = 1, \cdots, p; h = 1, \cdots, r),$$

has rank $p+r$. If a problem satisfies (H_1) and (H_2), and not (H_3), then the matrix (23) has rank $p+r-k$, where $0 < k \leqq p$. By deleting k of the end-conditions $\Psi_\gamma = 0$, $(\gamma = 1, \cdots, p)$, one may then obtain a problem which satisfies (H_3), and which is equivalent to the original problem in the sense that an arc η is admissible for this new problem if and only if it is admissible for the original problem. Such a modified problem will be called the normal boundary problem determined by the end-conditions $\Psi_\gamma = 0$, $(\gamma = 1, \cdots, p)$.

3. *Existence of Characteristic Values.* As indicated in the introduction, diverse methods of proof have been used in treating various special boundary problems of the general type formulated in the preceding section. This is particularly true of such boundary problems that are linear in the characteristic parameter. In this section we shall discuss the different methods that have been utilized in the proof of the existence of characteristic values. Instead of attempting to discuss the papers chronologically we shall group them according to the methods employed. Moreover, we shall discuss first those methods which have been used less within recent years in the consideration of boundary problems associated with the calculus of variations.

(a) *Application of the theory of integral equations.* As shown by Hilbert [1], a boundary value problem consisting of a single second order differential equation with real coefficients and corresponding self-adjoint boundary conditions is equivalent to an integral equation with a real symmetric closed kernel. Hence by Hilbert's theory of such integral equations the given boundary problem has an infinite number of real and no complex char-

acteristic values. The results of the papers of Cairns, Mason [1], and Myller [1] listed in the bibliography either depend directly upon, or are closely related to the Hilbert theory of integral equations. Marković ([1] and [2]) has recently applied the theory of integral equations to the study of periodic solutions of self-adjoint equations of even order with periodic coefficients.†

By the introduction of a Green's matrix the general boundary problem (17), (18) is reduced to the equivalent system of integral equations

$$\eta_i(x) = \lambda \int_{x_1}^{x_2} \kappa_{ij}(x, t)\eta_j(t)dt,$$

where $\kappa_{ij}(x, t) \equiv \kappa_{ji}(t, x)$. The relation of the linear system of the form (17), (18) to the more general system (21) will be discussed later.

(b) *Application of difference equations.* The method of difference equations has served as a heuristic approach to many kinds of transcendental problems. Indeed, it was the unpublished method which Sturm originally used in his investigations of boundary value problems. The rigorous passage to the limit from a system of difference equations to a differential system was carried through by Porter in 1902. Since then this method has been used by Bôcher, Fort, Carmichael, Richardson, Courant, Plancherel, W. M. Whyburn, and others in considering various types of boundary problems. We may, therefore, think of this method of passage to the limit as one of the well-established methods of treating such problems. If the speaker may venture a personal opinion, however, it seems that most results which have been established by the use of difference equations have subsequently been proved by other methods more elegant in detail.

(c) *Application of asymptotic expressions.* The use of asymptotic expressions for the characteristic values and corresponding characteristic functions, together with the use of these expressions in the theory of the development of arbitrary functions, dates from the work of Liouville. The reader is referred to

† For brevity, references to literature on special equations, such as those of Hill and Mathieu, are not included in the bibliography at the end of this paper.

Bôcher [1], pp. 177–179, for an introductory discussion of this method for a boundary problem consisting of a single second order differential equation and Sturmian boundary conditions. In more recent times, Birkhoff [1] has developed asymptotic expressions for the solutions of a single equation of the nth order involving a parameter in a general fashion, and has applied his results to a boundary problem which is linear in the characteristic parameter, and involves two-point boundary conditions. The coefficients of the differential equation and the boundary conditions of his problem are not assumed real; moreover, the system itself is not supposed to be self-adjoint. In certain general cases Birkhoff obtains the existence of infinitely many characteristic values, and a corresponding expansion theorem for arbitrary functions. In view of Birkhoff's general assumptions, it is not surprising that those real self-adjoint systems which come under the class considered by him may be treated by other methods which are simpler in detail than that involving the use of asymptotic expressions. All of the problems associated with the second variation of calculus of variations problems in the manner described in the preceding section are real and self-adjoint. Consequently we shall not consider in detail the results of Birkhoff, nor shall we discuss specifically the important subsequent work of Birkhoff, Tamarkin, Langer, Stone, and others in this field. For completeness, their papers are listed in the bibliography at the end of this paper. It is to be remarked in passing, however, that in the field of convergence and equiconvergence theorems no other method has as yet obtained as general theorems as those proved by the use of asymptotic expressions.

(d) *Application of linear algebraic equations in infinitely many variables.* In 1914 Lichtenstein [1] considered boundary value problems involving a single second order linear differential equation linear in the characteristic parameter, with associated Sturmian boundary conditions. By expanding admissible functions $\eta(x)$ in Fourier series, he showed that the characteristic values of the given problem were identical with the characteristic values of an infinite system of linear algebraic equations

$$(\delta_{ij} + \lambda\kappa_{ij})x_j = 0, \qquad (i, j = 1, 2, \cdots),$$

in the variables (x_i). The matrix $\|\kappa_{ij}\|$ is completely continuous

(vollstetig) in the sense of Hilbert. Under certain general conditions he proved the existence of infinitely many characteristic values for the given problem, and also established equi-convergence theorems in terms of the characteristic functions. Later, Lichtenstein [2] extended the method of proof to a system consisting of a second order integro-differential equation with the special boundary conditions $\eta(x_1)=0=\eta(x_2)$. Geiringer [1] has generalized the Lichtenstein method of proof to a system of differential equations of the second order, and Boerner [1] has used the method in considering a self-adjoint equation of the fourth order. Anna Pell Wheeler has applied the theory of linear algebraic equations in an infinite number of variables to a second-order differential equation with Sturmian boundary conditions. Her method is very intimately related to that of Lichtenstein, though not identical with it. Finally, E. Hölder has extended the Lichtenstein method to the boundary problem associated with the second variation of the problem of Lagrange with fixed end points, that is, to a system (17) with the boundary conditions $\eta_i(x_1)=0=\eta_i(x_2)$.

(e) *Application of general theorems on differential equations.* Bliss [2] considered a boundary problem of the form

$$
\begin{aligned}
&y_\sigma' = [\mathfrak{A}_{\sigma\tau}(x) + \lambda\mathfrak{B}_{\sigma\tau}(x)]y_\tau, && x_1 \leqq x \leqq x_2,\\
&\mathfrak{M}_{\sigma\tau}y_\tau(x_1) + \mathfrak{N}_{\sigma\tau}y_\tau(x_2) = 0, && (\sigma, \tau = 1, \cdots, \mathfrak{n}),
\end{aligned}
\tag{25}
$$

where the functions $\mathfrak{A}_{\sigma\tau}$, $\mathfrak{B}_{\sigma\tau}$ are real-valued and continuous on x_1x_2, and $\mathfrak{M}_{\sigma\tau}$, $\mathfrak{N}_{\sigma\tau}$ are real constants with the matrix $\|\mathfrak{M}_{\sigma\tau}; \mathfrak{N}_{\sigma\tau}\|$ of rank $\mathfrak{n}$. The problem adjoint to (25) is defined as

$$
\begin{aligned}
&z_\sigma' = -z_\tau[\mathfrak{A}_{\tau\sigma}(x) + \lambda\mathfrak{B}_{\tau\sigma}(x)],\\
&z_\tau(x_1)\mathfrak{P}_{\tau\sigma} + z_\tau(x_2)\mathfrak{Q}_{\tau\sigma} = 0,
\end{aligned}
\tag{26}
$$

where the coefficients $\mathfrak{P}_{\tau\sigma}$, $\mathfrak{Q}_{\tau\sigma}$ satisfy the relations

$$
\mathfrak{M}_{\sigma\nu}\mathfrak{P}_{\nu\tau} - \mathfrak{N}_{\sigma\nu}\mathfrak{Q}_{\nu\tau} = 0, \qquad (\sigma, \nu, \tau = 1, \cdots, \mathfrak{n}).
$$

This definition of adjoint system is equivalent to that originally introduced by Bounitzky [1]. Bliss called the system (25) self-adjoint if it is equivalent to (26) under a non-singular linear transformation $z_\sigma = \mathfrak{T}_{\sigma\tau}(x)y_\tau$. When a further hypothesis was satisfied the system was termed definitely self-adjoint. Bliss proved that a definitely self-adjoint system has only real char-

acteristic values, and that these values are denumerably infinite in number. Moreover, certain general expansion theorems were proved by methods analogous to those of the Hilbert-Schmidt integral equation theory.

The system (17), (18) is self-adjoint with respect to the transformation matrix

$$\|\mathfrak{T}\| \equiv \left\| \begin{matrix} 0_{ij} & \delta_{ij} \\ -\delta_{ij} & 0_{ij} \end{matrix} \right\|, \qquad (i, j = 1, \cdots, n).$$

This system, however, is definitely self-adjoint according to the definition of Bliss [2] only in case $|B_{ij}| \neq 0$, that is, in case there are no auxiliary differential equations $\Phi_\alpha = 0$. In this connection it is to be pointed out that Cope [1] makes the erroneous statement that the general system (17), (18) is definitely self-adjoint according to the definition of Bliss [2], and this incorrect statement has not been eliminated in the slightly changed form of his thesis which has only recently been published.

Hu [1] has considered specifically the system (17), (18), and by a method of proof similar to that of Bliss [2] has proved the existence of an infinity of characteristic values. Moreover, the existence theorem of Hu does not assume the strengthened Clebsch condition (H_2), but simply the non-singularity of the matrix (13). In a recent paper Bliss [5] has modified the definition of definite self-adjointness in such a way that the system (17), (18) is definitely self-adjoint according to this new definition. However, a general system (25) may be definitely self-adjoint according to this new definition and have no characteristic values or only a finite number. Reid [6] has shown that a system which is definitely self-adjoint according to this new definition, and which satisfies the additional restriction that the matrix $\|\mathfrak{B}_{\sigma\tau}\|$ is of constant rank on x_1x_2, is equivalent to a system of the form (22). The character of the equivalence of the given system to this latter system is new in the sense that the second system involves twice the number of dependent functions occurring in the original system. Conditions are given by Reid [6] for such a system to have an infinity of characteristic values. In particular, the problem considered by Hu is such a definitely self-adjoint system.

(f) *Application of the minimum principle.* Now consider a boundary problem (22) which satisfies hypotheses (H_1), (H_3),

for which the matrix (13) is non-singular, and such that the corresponding expression $\mathfrak{I}_2[\eta]$ is positive for all non-identically vanishing admissible arcs η. These conditions imply that the strengthened Clebsch condition (H_2) also holds. It is to be noted that whenever the form $\eta_i K_{ij}\eta_j$ is positive definite and (H_2) is satisfied, the hypothesis $\mathfrak{I}_2[\eta]>0$ for non-identically vanishing admissible arcs is not an essential restriction. In this case a value λ_0 exists such that if $\mathfrak{I}_2[\eta]$ is replaced by $\mathfrak{I}_2[\eta]+\lambda_0\int_{x_1}^{x_2}\eta_i K_{ij}\eta_j dx$ the modified quadratic functional satisfies the desired condition. The modified boundary problem is equivalent to the original problem by a linear change of parameter.

There will now be defined a sequence of classes of admissible arcs in which we shall consider the problem of minimizing the functional $\mathfrak{I}_2[\eta]$. The class S_1 is defined as the totality of admissible arcs satisfying the relation

$$\int_{x_1}^{x_2}\eta_i K_{ij}\eta_j dx = 1.$$

The sequence of classes is defined by induction as follows: suppose classes $S_1, \cdots, S_{s-1}$, $(s\geqq 2)$, have been defined and are not empty, and for $\lambda=\lambda_t$, $(t=1, \cdots, s-1)$, where λ_t is the greatest lower bound of $\mathfrak{I}_2[\eta]$ in the class S_t, there are r_t, $(0<r_t\leqq 2n)$, linearly independent solutions of (22). If $\eta_{i\kappa}, \zeta_{i\kappa}$, $(\kappa=1, 2, \cdots, r=r_1+\cdots+r_{s-1})$, denote these characteristic solutions, the class S_s is defined as the totality of arcs η of the class S_{s-1} which satisfy the relations

$$\int_{x_1}^{x_2}\eta_{i\kappa}K_{ij}\eta_j dx = 0, \qquad (\kappa = 1, \cdots, r).$$

Under the above hypotheses one may then prove the following theorem: *If the class S_s is non-vacuous and λ_s is the greatest lower bound of $\mathfrak{I}_2[\eta]$ in this class, then $\lambda=\lambda_s$ is a characteristic value of* (22) *and $\lambda_s>\lambda_{s-1}$.*

Various writers have proved for more or less general problems and by diverse methods the above theorem in the process of showing the existence of characteristic values. For the case of a single differential equation of the second order with self-adjoint boundary conditions Mason has used this method to prove the existence of characteristic values. The first proof of Mason [1]

depended upon the theory of integral equations, but in later papers (Mason [2], [3], and [4]) this proof was made independent of integral equation theory. A somewhat different proof of this result for a second order differential equation with Sturmian boundary conditions was given by Holmgren [1]. Reid [1] proved the above theorem for the general problem (22) by a method which closely parallels that of Mason.

Still more recently Reid [5], using methods analogous to those of Reid [1], has treated a type of integro-differential boundary problem which contains the differential boundary problems formulated in §2 as special cases. In Reid [5] there is first considered a self-adjoint integro-differential boundary problem related to the functional $\mathfrak{I}_2[\eta]+\int_{x_1}^{x_2}\int_{x_1}^{x_2}M_{ij}(x, t)\eta_i(x)\eta_j(t)dxdt$ in the manner that the problem (17), (18) is related to $\mathfrak{I}_2[\eta]$. Such a system reduces to (17), (18) for $M_{ij}(x, t)\equiv 0$. Under the assumption that the expressions $\mathfrak{I}_2[\eta]$, $\Phi_\alpha(x, \eta, \eta')$, $\Psi_\gamma[\eta(x_1), \eta(x_2)]$ satisfy hypotheses (H_1), (H_2), (H_3), while the functions $M_{ij}(x, t)$ are continuous and $M_{ij}(x, t)=M_{ji}(t, x)$, it is proved that this accessory integro-differential problem has infinitely many characteristic values. There is also considered in Reid [5] an integro-differential problem that in general involves the parameter λ non-linearly, and which reduces to the differential system (21) when certain functions are identically zero. The method of treatment is that of considering an auxiliary boundary problem which is linear in a second characteristic parameter Λ and related to the quadratic functional $\mathfrak{I}_2[\eta:\lambda]$ of the given problem in a manner analogous to that in which the differential system (21) is related to (20). The characteristic values of the given problem are then those values of λ for which some characteristic value of the auxiliary problem is equal to zero. Several sets of sufficient conditions for the existence of infinitely many characteristic values are given, and, when stated for the special problems of the form (21) previously considered by Morse and Birkhoff and Hestenes, these results go beyond those obtained by the authors just mentioned.

For the accessory boundary problem (17), (18), that is, the system (22) with $K_{ij}=\delta_{ij}$, the above theorem may be established readily by the use of theorems of the classical calculus of variations. This was first indicated in a short paper by Bliss [1] for a plane problem of the calculus of variations with variable

end points. He proved by calculus of variations methods the extremizing property of the smallest characteristic value of the associated boundary problem, and established the equivalence of the non-negativeness of the characteristic values and the usual Jacobi condition. Hickson [1] treated the accessory boundary problem for a fixed end-point problem of the calculus of variations in n-space by the use of classical theorems of the calculus of variations. Finally, this calculus of variations method has been extended to the general problem (17), (18) by Wiggin [1]. One might also establish the above theorem for the problem (22) by similar methods, but in view of the results of Reid [5], §7, one may readily deduce this result from the corresponding theorem for the accessory problem (17), (18).

Another type of proof of the above theorem is given by the existence theorems for the problem of Mayer that have been established by Graves [1].

One may also define the characteristic values by means of the maximum-minimum property introduced by Courant, which does not involve the use of characteristic functions. This method is particularly useful in the treatment of partial differential and more general functional equations.

For a discussion of the application of the method of Ritz to certain self-adjoint boundary problems the reader is referred to Kryloff [1]. This memoir contains a bibliography of the numerous publications of Kryloff and others on this topic.

(g) *Application of comparison and oscillation theorems.* As indicated in the introduction, Sturm utilized comparison and oscillation theorems in the proof of the existence of infinitely many characteristic values for the problem he considered. Bôcher greatly simplified the details and extended the results of the Sturmian theory. For a bibliography of the extensive publications of Bôcher the reader is referred to Birkhoff [4]. For brevity, Bôcher's papers are not listed separately in the bibliography of the present paper. Further results on boundary problems associated with a single second order differential equation have been proved by Ettlinger ([1] and [2]).

In his statement of sufficient conditions for the problem of Bolza, Morse [2] used the boundary value problem formulation of the Jacobi condition, established for this problem comparison and oscillation theorems, and showed the existence of infinitely

many characteristic values. In his Colloquium lecture Morse [3] developed more general separation, comparison, and oscillation theorems for problems which are in general non-linear in the parameter. Although his results are stated for the problem in n-space without auxiliary differential equations, the methods are seen to extend readily to the general problem of Bolza when suitable normality assumptions are made. From the standpoint of the existence of characteristic values we may say that Morse's method is that of comparison and oscillation theorems, since such theorems are first established and then used in the proof of the existence of infinitely many characteristic values. The comparison and oscillation theorems, together with his results on the existence of characteristic values, are obtained under hypotheses which are entirely analogous to those of the Sturmian theory. In particular, it is assumed that for arbitrary non-identically vanishing admissible arcs η the functional (20) is a proper monotone decreasing function of λ.

Birkhoff and Hestenes have also treated boundary value problems associated with problems of the calculus of variations involving no auxiliary differential equations. Their results on the existence of characteristic values, similar to those of Morse, are also obtained by the use of comparison and oscillation theorems. Further discussion of their results will be given in the next section on comparison and oscillation theorems.

4. *Oscillation and Comparison Theorems.* In the introduction we have indicated the general character of the comparison and oscillation theorems of the Sturmian theory. The present section is concerned with the great extensions and generalizations of these theorems that have been established within recent years. Before taking up this more recent literature, however, we note that for a second order equation linear in the parameter and having associated Sturmian boundary conditions Richardson [1] proved the oscillation properties of the characteristic functions using the Jacobi condition of the calculus of variations and the extremizing properties of the characteristic values. This last property was deduced from integral equation theory.

As indicated in §3(b), various writers have reconstructed and extended Sturm's work on second-order difference equations, proving oscillation and comparison theorems for such equations. Results for differential systems have been obtained by limiting

processes from corresponding results for difference equations by Porter, Bôcher, Whyburn, and others. The papers [2] and [3] of Carmichael contain a highly interesting treatment of difference equations, together with a discussion of the intimate relationships that exist between algebraic and transcendental problems.

The boundary problem for which Morse ([1] and [3]) states his general comparison and oscillation theorems is equivalent to a problem (21) *which does not involve any auxiliary differential equations* $\Phi_\alpha = 0$. It is assumed that for each value of λ hypothesis (H_2) is satisfied, and for λ sufficiently small the functional (20) is positive for arbitrary non-identically vanishing admissible arcs η; moreover, conditions are imposed which insure that for non-identically vanishing admissible arcs η the functional $\mathfrak{J}_2[\eta:\lambda]$ is a properly monotone decreasing function of λ. For such a problem (H_3) is satisfied whenever the matrix $\|\Psi_{\gamma;i1}\Psi_{\gamma;i2}\|$ is of rank p. By the use of broken extremals, Morse introduces a quadratic form in a finite number of variables whose coefficients depend upon λ and which has the following property: the form is singular if and only if λ is a characteristic value of (21); moreover, for such a value the nullity of the form is the index of λ as a characteristic value, and the negative type number of the form is equal to the number of characteristic values of (21) which are less than this value.

Suppose that B is the problem determined by Q, ω, Ψ_γ, $(\gamma = 1, \cdots, p)$, B^* is a second problem determined by Q^*, ω^*, Ψ_ν^*, $(\nu = 1, \cdots, p^*)$, and that each of these problems satisfies the above described conditions. We shall denote the characteristic values of B and B^* by $\{\lambda_s\}$ and $\{\lambda_s^*\}$, $(s = 1, 2, \cdots)$, respectively, whenever these values exist. If the relations $\Psi_\gamma[\eta(x_1), \eta(x_2)]$ are linearly dependent on those of the set $\Psi_\nu^*[\eta(x_1), \eta(x_2)]$, while $Q \equiv Q^*$, $\omega \equiv \omega^*$, Morse has called B^* a sub-problem of B of dimension $p^* - p$.

In case B and B^* have $\omega \equiv \omega^*$ Morse has proved comparison theorems of the following nature: (α) *comparison of a problem with a sub-problem*; (β) *comparison of two problems which differ only in the end form* Q; (γ) *comparison of two general problems.*

The essential result of type (α) may be stated as follows: *Suppose* B^* *is a sub-problem of* B *of dimension* $p^* - p$. *Then if* $\lambda_1, \cdots, \lambda_{s+p^*-p}$ *exist for* B *the characteristic values* $\lambda_1^*, \cdots, \lambda_s^*$ *exist for* B^* *and* $\lambda_{s+p^*-p} \geqq \lambda_s^* \geqq \lambda_s$, $(s = 1, 2, \cdots)$.

In case (β) let N and P denote, respectively, the negative and positive type numbers of the quadratic form Q^*-Q on the linear set of values $[\eta_i(x_1), \eta_i(x_2)]$ satisfying $\Psi_\gamma=0$. *Then if* $\lambda_1, \cdots, \lambda_{s+P}$ *exist for* B, *the characteristic values* $\lambda_1^*, \cdots, \lambda_s^*$ *exist for* B^* *and* $\lambda_{s+P} \geqq \lambda_s^*$; *if* $\lambda_1^*, \cdots, \lambda_{s+N}^*$ *exist for* B^*, *then* $\lambda_1, \cdots, \lambda_s$ *exist for* B *and* $\lambda_{s+N}^* \geqq \lambda_s$.

In the general case (γ) let $\Psi_\rho^0=0$, $(\rho=1, \cdots, p^0)$, be linearly independent conditions such that the problem B_0 involving Q, ω, Ψ_ρ^0 is a sub-problem of B, and the problem B_0^* involving Q^*, ω, Ψ_ρ^0 is a sub-problem of B^*. The results of Morse in case (γ) are obtained by applying the results of the previous cases to the problems B and B_0, B^* and B_0^*, B_0 and B_0^*. In particular, if B and B^* are two general problems the number of characteristic values of B on a given finite interval of the λ-axis differs from the number of characteristic values of B^* on this interval by at most $2n$.

For the case of different integrand functions ω we shall state only the following result: If $\mathfrak{I}_2^*[\eta:\lambda]$ *and* $\mathfrak{I}_2[\eta:\lambda]$ *denote the expressions* (20) *for* B^* *and* B, *respectively, and* $\mathfrak{I}_2^*[\eta:\lambda] \geqq \mathfrak{I}_2[\eta:\lambda]$ *for arbitrary non-identically vanishing admissible arcs* η, *then if* $\lambda_1^*, \cdots, \lambda_s^*$ *exist for* B^* *the characteristic values* $\lambda_1, \cdots, \lambda_s$ *exist for* B *and* $\lambda_s^* \geqq \lambda_s$. *If the strict inequality sign holds in the first relation, then the strict inequality sign also holds in the second relation.*

The general oscillation theorem of Morse [3] is as follows: If $v(\lambda)$ is the number of characteristic values of B less than λ, and $c(\lambda)$ is the number of points on $x_1<x<x_2$ conjugate to x_1 for this value of λ, then $v(\lambda)-(2n-p) \leqq c(\lambda) \leqq v(\lambda)$.

Birkhoff and Hestenes have also obtained comparison and oscillation theorems for the same problem that has been treated by Morse [3]. By the introduction of so called natural isoperimetric conditions for the functional $\mathfrak{I}_2[\eta]$, they are led to a characterization of the solutions of the corresponding equations $J_i(\eta)=0$ which satisfy the boundary conditions (17). In particular, a set of natural isoperimetric conditions is called a minimal set if $\mathfrak{I}_2[\eta] \geqq 0$ for every admissible arc η satisfying these conditions, and if no proper subset of these conditions has this property. The number of conditions in a minimal set is always the same, and this number is called the type number of $\mathfrak{I}_2[\eta]$ relative to the conditions $\Psi_\gamma=0$. For a non-singular problem this

type number is finite if and only if hypothesis (H_2) holds. In addition, Birkhoff and Hestenes introduce the term order of degeneracy for the maximum number of linearly independent solutions of $J_i[\eta]=0$ satisfying the boundary conditions (18). If (H_2) is satisfied, the type number and order of degeneracy of $\mathfrak{J}_2[\eta]$ relative to $\Psi_\gamma=0$ are equal respectively to the negative type number and the nullity of the Morse quadratic form. The oscillation and comparison theorems of Birkhoff and Hestenes are somewhat more complete than those of Morse, because of the introduction of the notion of order of concavity of a problem B with respect to a sub-problem B^*. Hestenes [3] has recently announced the extension of these results to the general boundary problem associated with the problem of Bolza as formulated in §2.

Because of their generality, we have first described the comparison and oscillation theorems obtained by Morse and Birkhoff and Hestenes for the general problem (21). It is to be noted, however, that for the problem (17), (18) which involves the parameter linearly these comparison and oscillation theorems are deducible from the extremizing properties of the characteristic values. Hu [1] has used this method in proving certain comparison and oscillation theorems for the accessory problem. This method has also been used by Reid [5] in establishing general comparison and oscillation theorems for the integro-differential problem there considered. Moreover, as pointed out in the preceding section, Reid has shown that the theory for the general boundary problem (21) may be made to depend upon an associated boundary problem involving a second parameter linearly. Comparison and oscillation theorems for the general problem are ready consequences of the corresponding theorems for the associated problem. This method of treating the general problem non-linear in the parameter seems to be of distinct interest because of the diverse methods of proof that are available for the treatment of the associated problem linear in the second parameter. It is to be noted that in order to establish an oscillation theorem for the general problem (21) when no additional assumption of normality is made it is necessary to formulate a new definition of conjugate points. The definition given by Reid [5] for the integro-differential problem there treated reduces for the differential problem (21) to the follow-

ing: For $x_1<c<x_2$ let B_c denote the normal boundary problem determined by the functional $\mathfrak{I}_2[\eta:\lambda]$ and the end-conditions $\eta_i(x_1)=0=\eta_i(c)$. The value c is said to be a conjugate point of $x=x_1$ on x_1x_2 relative to the differential equations of (21) for $\lambda=\lambda_0$ if: (i) $\lambda=\lambda_0$ is a characteristic value of B_c; (ii) there is at least one characteristic solution $\eta_i=y_i$, $\zeta_i=z_i$ of B_c such that for arbitrary ϵ satisfying $0<\epsilon<x_2-c$ there is no corresponding set of functions $\zeta_i(x)$ forming with the arc $\eta_i=y_i(x)$ on x_1c, $\eta_i\equiv 0$ on $c\leqq x\leqq c+\epsilon$ a characteristic solution of $B_{c+\epsilon}$ for $\lambda=\lambda_0$. It is easily seen that this definition of a conjugate point reduces to the older definition when certain additional assumptions of normality are satisfied.

5. *Application of Boundary Value Problems to Sufficiency Theorems in the Calculus of Variations.* As indicated in the introduction for a simple problem of the calculus of variations, the Jacobi condition may be phrased in terms of a boundary value problem. For a detailed discussion of this approach for a simple problem the reader is referred to chapter VI of Lovitt [1]. Lichtenstein [3] and [4] has used the method of boundary problems in obtaining sufficient conditions for a weak relative minimum in certain problems of the calculus of variations. Boerner [2] and [3] has applied the method of Lichtenstein to problems involving higher derivatives and to the parametric problem in the plane with fixed end-points. As indicated in §3(f), Bliss [1] has expressed the Jacobi condition for a plane calculus of variations problem with variable end-points in terms of the characteristic values of a boundary problem. For the use of this formulation of the Jacobi condition in connection with sufficiency theorems for more general problems of the calculus of variations, the reader is referred to Morse [2], Myers [1], Hestenes [1] and [2], Bliss [4], and Reid [3] and [4]. A comparison of different formulations of the Jacobi condition has been made by Reid [2]. It is of interest to note that the boundary problem used by Reid [3] in treating discontinuous solutions in the non-parametric problem involves boundary conditions at more than two points. By a very simple transformation, however, this problem is reducible to one of the form (17), (18) with two-point boundary conditions.

6. *Remarks.* We shall now discuss briefly certain aspects of

the theory of boundary problems of the type formulated in §2 that are in need of further consideration. It follows readily that a real self-adjoint boundary problem consisting of a single differential equation of even order and associated two-point boundary conditions may be reduced to a problem of the sort formulated in §2. Thus, for example, all the theorems of Janczewsky [1] and Cimmino [1] on the existence of characteristic values for a real self-adjoint fourth order boundary problem follow readily from the theory of the general problem of this paper. However, from the oscillation theorems that have been established for this general problem one does not obtain all of the oscillation theorems that have been proved for real self-adjoint boundary problems whose differential equations are of even order greater than two. As yet no one has made an exhaustive study of the relations between comparison and oscillation theorems for such a system and the corresponding theorems for the general problem of §2.

In all of the above discussion of the general problems of §2 we have assumed that the matrix (13) is non-singular. We do not have time in this lecture to consider the work of Hilbert, Weyl, and others concerning singular boundary value problems. We wish to mention, however, a recent paper by Morse and Leighton on singular quadratic functions because their method of attack is that of the calculus of variations. It would be highly desirable to extend their work to singular quadratic functionals involving more than a single dependent function, and to consider the theory of associated boundary value problems.

As indicated in §3(c), various authors have obtained by the use of asymptotic expansions stronger results for the expansion problem than have been obtained by other methods. The question of the convergence and summability properties of the generalized Fourier expansions in terms of the characteristic functions of the general boundary problem here considered is a fertile field for further research.

There is also the question of boundary problems of the type formulated in §2 which involve more than a single characteristic parameter. Hilbert [1], Yoshikawa [1], [2], Hilb [1], Richardson [2], [3], and [4], and others have considered the question of a Klein oscillation theorem for boundary problems associated with second order differential equations. To date this

question has not been considered in detail for more general problems of the type here discussed. A start in this direction has been made by Reid [7].

Another field of interest is the consideration of the character of solutions of the boundary problem when the independent variable x is complex. Hille has considered rather extensively the question of oscillation theorems for second order differential equations in the complex domain. So far as I know, however, such a study has not been made of more general problems of the sort considered in this paper.

Finally, the question of the use of boundary problems in considering more general functionals of the calculus of variations is of distinct importance. In his recent Chicago dissertation Goldstine [1] has considered a somewhat general functional of lines, and has formulated an analogue of the Jacobi condition in terms of an integral equation. It is of interest to note that for the classical calculus of variations problem his formulation of the Jacobi condition is expressible in terms of a boundary problem of the fourth order.

7. *Boundary Problems Associated with Multiple Integrals of the Calculus of Variations.* In concluding this address, I wish to speak very briefly concerning boundary value problems associated with double and multiple integrals of the calculus of variations. For a double integral problem involving a single dependent function the Jacobi equation along an extremal which satisfies the strengthened Legendre condition is an elliptic partial differential equation of the second order. Consequently all the literature concerning such partial differential equations may be thought of as concerning the Jacobi equation for a calculus of variations problem. We shall, however, restrict our comments here to the discussion of a few papers which are specifically concerned with the phrasing of the Jacobi condition for double and multiple integrals in terms of a boundary problem associated with the Jacobi equation.

Lichtenstein [7], [8], and [9] has utilized such a formulation of the Jacobi condition for double integral problems with fixed boundary and also for such problems involving variable boundary conditions of a very simple type. He assumed that the integrand function was analytic, and by the use of expansion theorems in terms of the characteristic functions of the bound-

ary problem he proved that when the smallest characteristic number of his problem was greater than unity the given extremal surface afforded the integral a weak relative minimum Picone [2] has also expressed the Jacobi condition in terms of the characteristic values of a boundary problem. Reid [8] established by use of methods of the calculus of variations the minimizing property of the least characteristic number of the associated boundary problem, and obtained a corresponding proof of sufficient conditions for a weak relative minimum for the double integral problem with fixed boundary values. Reid [9] has given an extended form of the Jacobi condition which is of use in considering the associated boundary value problem when the integrand function and the given extremal are not assumed to be analytic.

For double and n-tuple integrals of the calculus of variations involving a single dependent function Simmons ([1] and [2]) has formulated an analogue of the Jacobi condition in terms of a boundary problem. One may readily formulate such an analogue for still more general multiple integrals of the calculus of variations.

Using the theory of boundary value problems, Birkhoff and Hestenes have proved the existence of a set of natural isoperimetric conditions for the second variation along an extremal of a double integral problem which satisfies the strengthened Legendre condition.

It is to be remarked that for these accessory problems associated with double and multiple integral problems one may prove certain comparison theorems; however, the results which have been obtained concerning oscillation theorems for such problems are quite meager.

The properties of minimizing surfaces for double and multiple integral problems which are not extremals in the usual sense, that is, which are not of class C^2, have been studied by Haar and his associates (see Haar [1]; also Bliss [7]). For such a minimizing surface the usual Jacobi equation is replaced by a pair of partial differential equations. One may readily set up a corresponding boundary problem and state that as a necessary condition for a minimum this problem can have no negative characteristic values. However, the question of the existence and the character of the solutions of such a boundary value

problem has not yet been solved. The study of this sort of boundary problem and its application to the proof of sufficient conditions for an extremum in such a problem is one of distinct interest and importance in the calculus of variations.

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I. BOUNDARY PROBLEMS ASSOCIATED WITH SIMPLE INTEGRALS OF THE CALCULUS OF VARIATIONS

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II. BOUNDARY PROBLEMS ASSOCIATED WITH DOUBLE AND MULTIPLE INTEGRALS OF THE CALCULUS OF VARIATIONS

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THE UNIVERSITY OF CHICAGO

p-ALGEBRAS OF EXPONENT p*

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A. A. Albert and O. Teichmüller have recently investigated the structure of p-algebras, that is, normal simple algebras of degree p^e and characteristic p.‡ In particular they showed that a necessary and sufficient condition that such an algebra have exponent p is that it be similar to an algebra A having a maximal subfield $C=F(c_1, c_2, \cdots, c_m)$, where $c_i^p=\gamma_i \epsilon F$, the underlying field. The latter algebra is cyclic. It is the purpose of this note to apply some results of my paper *Abstract derivation and Lie algebras*§ to obtain a new generation of A. For $m=1$ this generation is more symmetric than the cyclic generation. We obtain a condition that A be a matrix algebra in terms of the new generation, and when $m=1$ we have as a consequence a reciprocity law for fields of characteristic p.

Let A be a normal simple algebra of degree p^m (order p^{2m}) over a field F of characteristic p and suppose A contains the maximal subfield $C=F(c_1, c_2, \cdots, c_m)$, $c_i^p=\gamma_i \epsilon F$. Let D be an arbitrary derivation of C over F, that is, a mapping $x \to xD$ of C into itself such that

$$(x+y)D = xD + yD, \qquad (x\alpha)D = (xD)\alpha,$$
$$(xy)D = (xD)y + x(yD), \qquad \alpha \epsilon F.$$

It is known that D may be chosen so that the only elements z such that $zD=0$ are those of F,‖ and for a D of this type I have shown that

$$(1) \qquad x(D^{p^m} + D^{p^{m-1}}\tau_1 + \cdots + D\tau_m) = 0, \qquad \tau_i \epsilon F,$$

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‡ A. A. Albert, *On normal division algebras of degree p^e over F of characteristic p*, Transactions of this Society, vol. 39 (1936), pp. 183–188, and *Simple algebras of degree p^e over a centrum of characteristic p*, Transactions of this Society, vol. 40 (1936), pp. 112–126. O. Teichmüller, *p Algebren*, Deutsche Mathematik, vol. 1 (1936), pp. 362–388.

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or

$$x^{(p^m)} + x^{(p^{m-1})}\tau_1 + \cdots + x'\tau_m = 0$$

for all $x \in C$, but no equation of the form

$$x^{(r)} + x^{(r-1)}b_1 + \cdots + x'b_{r-1} + b_r = 0, \qquad b_i \in C,$$

can hold if $r < p^m$.* I have shown also that any derivation in a simple subalgebra of a normal simple algebra may be extended to an inner derivation in the latter.† Thus there exists an element d in A such that $[x, d] \equiv xd - dx = xD$ for all $x \in C$.

We note that

$$xd^k = d^k x + C_{k,1}d^{k-1}x' + \cdots + x^{(k)}, \tag{2}$$

where the coefficients are those of the binomial theorem, and hence $xd^{p^j} = d^{p^j}x + x^{(p^j)}$. It follows from (1) that $(d^{p^m} + d^{p^{m-1}}\tau_1 + \cdots + d\tau_m)$ commutes with every x, and since C is a maximal subfield of A, $(d^{p^m} + d^{p^{m-1}}\tau_1 + \cdots + d\tau_m) = c \in C$. Deriving with respect to d (taking commutators), we have $[c, d] = 0$, and so $c = \delta \in F$ and

$$d^{p^m} + d^{p^{m-1}}\tau_1 + \cdots + d\tau_m = \delta. \tag{3}$$

We assert that C and d generate the whole of A. Suppose

$$d^r + d^{r-1}b_1 + \cdots + b_r = 0, \qquad b_i \in C, \tag{4}$$

is an equation of least degree having coefficients in C and satisfied by d. If $x \in C$ by (2)

$$d^{r-1}x_1 + d^{r-2}x_2 + \cdots + x_r = 0,$$

where, if we use the $C_{r,k}$ notation for binomial coefficients,

$$x_1 = C_{r,1}x', \qquad x_2 = C_{r,2}x'' + C_{r-1,1}x'b_1, \cdots,$$
$$x_r = x^{(r)} + x^{(r-1)}b_1 + \cdots + x'b_{r-1}.$$

Since (4) has minimum degree, $x_1 = x_2 = \cdots = x_r = 0$. But by (1) $x_r = 0$ is impossible for all x unless $r \geqq p^m$. It follows that $r = p^m$ and $1, d, \cdots, d^{p^m-1}$ are (right) independent over C. Thus C and d generate an algebra of order p^m over C and hence p^{2m} over F, and so C and d generate all of A. The field C, the derivation

* J, p. 218.
† J, p. 214.

D, and the equation (3) give a complete description of A.

Let $V(x)$ for $x\epsilon C$ be the function

$$V_{p^m}(x) + V_{p^{m-1}}(x)\tau_1 + \cdots + V_1(x)\tau_m,$$

where

$$V_{p^j}(x) = x^{p^j} + (x^{(p-1)})^{p^{j-1}} + (x^{(p^2-1)})^{p^{j-2}} + \cdots + x^{(p^j-1)}.$$

I have shown that $V(x)\epsilon F$, and that it has properties analogous to the norm in cyclic fields.* Now suppose $\delta = V(x_0)$. If $d_1 = d - x_0$, then $[x, d_1] = xD$ for all x, and since

$$d_1^{p^j} = (d - x_0)^{p^j} = d^{p^j} - V_{p^j}(x_0),$$

we have

$$d_1^{p^m} + d_1^{p^{m-1}}\tau_1 + \cdots + d_1\tau_m = 0,$$

and so $A \cong F_{p^m}$, the algebra of all p^m-rowed square matrices with elements in F.†

Conversely suppose that $A \cong F_{p^m}$. Then there exists in A a field $\tilde{C} \cong C$ and an element $\tilde{d}_1$ such that $[\tilde{x}, \tilde{d}_1] = \widetilde{xD}$ where $x \longleftrightarrow \tilde{x}$ in the isomorphism between C and $\tilde{C}$ and

$$\tilde{d}_1^{p^m} + \tilde{d}_1^{p^{m-1}}\tau_1 + \cdots + \tilde{d}_1\tau_m = 0.$$

This isomorphism between C and $\tilde{C}$ may be extended to an automorphism in A.‡ Hence there exists an element d_1 corresponding to $\tilde{d}_1$ such that $[x, d_1] = xD$ and

$$d_1^{p^m} + d_1^{p^{m-1}}\tau_1 + \cdots + d_1\tau_m = 0.$$

We observe that $d - d_1$ commutes with all the elements of C, and hence $d_1 = d - x_0$, $x_0 \epsilon C$. It follows as before that $\delta = V(x_0)$.

THEOREM. *A necessary and sufficient condition that A be $\cong F_{p^m}$ is that $\delta = V(x_0)$, $x_0 \epsilon C$.*

We now consider the special case where $m = 1$, $C = F(c)$, $c^p = \gamma$. Let D be the derivation such that $cD = 1$. It is easily seen that $D^p = 0$ and hence A is generated by c and d such that

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* See J, p. 224.

† The symbol $\cong$ denotes isomorphism. For the above equations and result see J, p. 223.

‡ M. Deuring, *Algebren*, 1935, p. 42.

$[c, d]=1$ and $d^p=\delta$. Thus A has the basis d^ic^j $(i, j=0, 1, \cdots, p-1)$ such that

$$c^p = \gamma, \qquad d^p = \delta, \qquad cd - dc = 1. \tag{5}$$

The condition that $A \cong F_p$ is $\delta = V(x_0)$, $x_0 \in F(c)$. Here $V(x)=x^p+x^{(p-1)}$, and so if $x=\xi_0+c\xi_1+\cdots+c^{p-1}\xi_{p-1}$, then

$$V(x) = (\xi_0^p - \xi_{p-1}) + \gamma\xi_1^p + \cdots + \gamma^{p-1}\xi_{p-1}^p. \tag{6}$$

If δ is not a p-th power, (5) is essentially symmetric in c and d. We define the derivation $d \to dE = -1$ in $F(d)$. Since $E^p=0$, the condition that $A \cong F_p$ is that

$$\gamma = V(y_0) = y_0E^{p-1} + y_0^p, \quad y_0 \in F(d).$$

But if

$$y = \eta_0 + d\eta_1 + \cdots + d^{p-1}\eta_{p-1},$$

then

$$V(y) = (\eta_0^p - \eta_{p-1}) + \delta\eta_1^p + \cdots + \delta^{p-1}\eta_{p-1}^p.$$

Thus we have the following reciprocity theorem for arbitrary fields of characteristic p.

THEOREM. *If γ and δ are not p-th powers in F, then $(\xi_0^p-\xi_{p-1})+\gamma\xi_1^p+\cdots+\gamma^{p-1}\xi_{p-1}^p=\delta$ is solvable for $\xi_i \in F$ if and only if $(\eta_0^p-\eta_{p-1})+\delta\eta_1^p+\cdots+\delta^{p-1}\eta_{p-1}^p=\gamma$ is solvable for $\eta_i \in F$.*

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CONCERNING NORMAL AND COMPLETELY NORMAL SPACES*

BY F. B. JONES

Urysohn has shown that any completely separable, normal topological space is metric. It is the principal object of this paper to establish a similar result for certain separable spaces.

THEOREM 1. *Every subset of power c† of a separable normal‡ Fréchet space-L (or -H) has a limit point.*

PROOF. Suppose, on the contrary, that S is a separable, normal Fréchet space-L (or -H) which contains a point set M of power c having no limit point. Let Z denote a countable subset of S such that every point of S either belongs to Z or is a limit point of Z. Since S is normal, there exists for each proper subset J of M a domain D_J which contains J but which neither contains a point of $M-J$ nor has a limit point in $M-J$. If J and K are two different proper subsets of M, then $Z \cdot D_J$ and $Z \cdot D_K$ are different subsets of Z. Hence, there are at least as many subsets of Z as there are proper subsets of M. However, since M is of power c and Z is only countable, there are *more* than c proper subsets of M but at most c subsets of Z. This is a contradiction.

The above argument with slight changes establishes the following three theorems.

THEOREM 2. *Every subset of power c of a separable, completely normal§ Fréchet space-L (or -H) contains a limit point of itself.*

THEOREM 3. *If $2^{\aleph_1} > 2^{\aleph_0}$, every uncountable subset of a separable normal Fréchet space-L (or -H) has a limit point.*‖

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* Presented to the Society, October 28, 1933.

† The number c is the power of the continuum.

‡ A space is said to be *normal* provided that, if P and Q are two mutually exclusive closed sets, there exist two mutually exclusive domains containing P and Q respectively.

§ A space is said to be *completely normal* provided that, if P and Q are two mutually separate point sets, there exist two mutually exclusive domains containing P and Q respectively.

‖ The numbers $\aleph_0$ and $\aleph_1$ are the first and second transfinite cardinals respectively. That $2^{\aleph_1} > 2^{\aleph_0}$ is an immediate consequence of a well known theorem if the *hypothesis of the continuum* holds true, that is, if $\aleph_1 = c$.

THEOREM 4. *If* $2^{\aleph_1} > 2^{\aleph_0}$, *every uncountable subset of a separable completely normal Fréchet space-L (or -H) contains a limit point of itself.*

A space-L may, however, be separable and normal but contain an uncountable point set *not* containing a limit point of itself. This is shown by the following example. A lemma will be established first.

LEMMA A. *There exists on the number interval* I(0, 1) *a point set* M *of power* $\aleph_1$ *such that if* K *is a countable subset of* M, *then* K *is an inner limiting set with respect to* M.*

PROOF. Let α denote a well-ordered sequence whose elements are the points of I(0, 1). Let P_1 denote the first point of α. Let Q_2 denote an inner limiting set of I(0, 1) of measure zero containing P_1, and let P_2 denote the first point of α in $M-Q_2$. Let Q_3 denote an inner limiting set of measure zero containing $P_1+P_2+Q_2$. Let P_3 denote the first point of α in $M-Q_3, \cdots$. In general, if $\bar{z}$ is an ordinal less than ω_1 and for each ordinal z, $1<z<\bar{z}$, P_z and Q_z are defined, then $\sum(P_z+Q_z)$, $z<\bar{z}$, is of measure zero and there exists an inner limiting set $Q_{\bar{z}}$ of I(0, 1) of measure zero containing $\sum(P_z+Q_z)$, $z<\bar{z}$. Now I(0, 1) $-Q_{\bar{z}}$ is of power c. Let $P_{\bar{z}}$ denote the first point of α in I(0, 1) $-Q_{\bar{z}}$. Let β denote the sequence $P_1, P_2, P_3, \cdots, P_z, \cdots$ and let M denote the subset of I(0, 1) whose points are the elements of β.

It is evident from the construction that M is of power $\aleph_1$. Suppose that K is a countable subset of M. Let $P_{\bar{z}}$ denote the first point of M in β which follows K in β and let $H=\sum P_z$, $z<\bar{z}$. Then K is a subset of H, which is clearly an inner limiting set with respect to M. But $H-K$ is countable since both H and K are countable, and therefore K is an inner limiting set with respect to M.

AN EXAMPLE. Let M' denote a subset of I(0, 1) of power $\aleph_1$ such that every countable subset of M' is an inner limiting set with respect to M' and let Z' denote the set of all points (x, y) of the plane such that (1) both x and y are rational numbers, (2) $0<x<1$, and (3) $y>0$. Furthermore, let α denote a well ordered sequence of the points of M' such that if P is a point

* The symbolism G_δ is used by some to denote an *inner limiting set.*

of M', it belongs to α and is preceded in α by only a countable subset of M'. Let S denote a space consisting of the points of M' and Z' in which *sequential limit point* is defined as follows: I. A point P of Z' is the sequential limit point of a sequence of points $P_1, P_2, P_3, \cdots$ of S provided there exists a number N such that if $n>N$, then $P_n=P$. II. A point P of M' is the sequential limit point of a sequence of points $P_1, P_2, P_3, \cdots$ of Z' provided that (1) it is the sequential limit point of $P_1, P_2, P_3, \cdots$ in the plane and (2) the line PP_n approaches the normal to I(0, 1) at P as n increases without limit. III. A point P of M' is the sequential limit point of a sequence of points $P_1, P_2, P_3, \cdots$ of M' provided that for each element a of α preceding P there exists an integer N such that if $n>N$, then either (1) $P_n=P$ or (2) P_n is between a and P in α; IV. In general, a sequence of points $P_1, P_2, P_3, \cdots$ of S has a sequential limit point P provided that P is by I, II, and III the sequential limit point of each of its subsequences which lie in M' or Z'. In order that it may be easier to keep in mind which limit point notion is being used, we shall adopt the convention that if H is a subset of S, H shall denote the point set as a subset of S and H' shall denote the corresponding subset of the plane.

From the definition of the above paragraph it is easy to see that S is a separable Fréchet space-L. It will now be shown that S is normal.

Suppose that H and K are two mutually exclusive closed subsets of S. If both are uncountable, then $H \cdot M$ and $K \cdot M$ are uncountable, and letting A_1 denote the first point of H in α, B_1 denote the first point of K which follows A_1 in α, A_2 denote the first point of H which follows B_1 in $\alpha, \cdots$, we see that $A_1, A_2, A_3, \cdots$ and $B_1, B_2, B_3, \cdots$ have the same sequential limit point. But since the sets are closed and mutually exclusive, this is impossible. Consequently one of the two sets is countable. We shall suppose that K denotes the countable set.

Since no point of Z is a sequential limit point of any sequence of distinct points of S, any subset of Z is a domain. Hence if H is a subset of Z, $D_H=H$ and $D_K=S-H$ are two mutually exclusive domains containing H and K respectively. Likewise if K is a subset of Z, then $D_H=S-K$ and $D_K=K$ are two mutually exclusive domains containing H and K respectively.

On the other hand, if H and K contain points of M, then for

each point P of $K \cdot M$, let a denote the first point of α which precedes P in α such that no point of $H \cdot M$ is between a and P in α. Let d_P denote P together with all points of M which are between a and P in α. Then d_P is a closed domain with respect to M containing P, and $D_{1K} = \sum d_P$ is a domain with respect to M covering $K \cdot M$ and containing no point of H. Furthermore, D_{1K} is closed, for if C were a limit point of $\sum d_P$ not belonging to $\sum d_P$, then it would belong to M and be a limit point of the points P of $K \cdot M$, and hence belong to $K \cdot M$. Since D_{1K} is countable, let $D_{1K} = O_1 + O_2 + O_3 + \cdots$. Then $D'_{1K} = O'_1 + O'_2 + O'_3 + \cdots$ is an inner limiting set with respect to M'. It shall be assumed for convenience that D'_{1K} does not contain the end points of $I(0, 1)$. There exists a sequence of point sets $D'_1, D'_2, D'_3, \cdots$ of $I(0, 1)$ such that (1) for each n, D'_n contains D'_{n+1}, (2) the common part of $D'_1 \cdot M, D'_2 \cdot M, D'_3 \cdot M, \cdots$ is D'_{1K}, and (3) for each n, D'_n is the sum of a set of non-overlapping segments $d'_{1n}, d'_{2n}, d'_{3n}, \cdots$. For each segment d'_{jn}, let r'_{jn} denote the interior of a regular hexagon in the plane having d_{jn} as a diameter and let $R'_n = \sum r'_{jn}$. For each n, let t'_n denote the interior of an inverted equilateral triangle lying in R'_n with its base parallel to the X-axis and lower vertex at O'_n. Let T' denote the set of all points X' and Z' such that, for some n, X' is in t'_n. Now $D_K = D_{1K} + (T - T \cdot H)$ is a domain with respect to S. Furthermore, no point of $M - D_{1K}$ is a limit point of D_K. For if P is a point of $M - D_{1K}$, P is not a limit point of D_{1K} and there exists an integer k such that R'_k does not contain P'. But no sequence of points $P'_1, P'_2, P'_3, \cdots$ lying in T' has P' as a sequential limit point in the plane such that the line $P'P'_n$ approaches the normal to the X-axis at P' since, for each n, $P'P'_n$ would make an angle of at least 30° with this normal when P'_n lies in R'_k. Hence D_K is closed and contains K but no points of H. Therefore, $D_H = S - D_K$ and D_K are mutually exclusive domains containing H and K respectively, and S is normal.

The reader will observe that if N is an uncountable subset of S, then $N \cdot M$ is uncountable and N has a limit point, namely, the first point P of α such that infinitely many points of M precede P in α. But it is clear that not every uncountable subset of S *contains* one of its limit points; for suppose that N is the set of all points P of M such that there is a first point of M in α pre-

ceding P in α. Then N is uncountable and contains none of its limit points.

In order to make an application of Theorem 4, two lemmas will be established. Throughout the rest of this paper M denotes a space satisfying Axiom 0 and parts 1, 2, and 3 of Axiom 1 of R. L. Moore's *Foundations of Point Set Theory* and is referred to as a *Moore space M*.

DEFINITION. A space is said to have the *Lindelöf property* provided that if G is a collection of domains of the space covering a point set K, then G contains a countable subcollection G' covering K.

LEMMA B. *In order that a Moore space M should have the Lindelöf property it is necessary and sufficient that every uncountable subset of M should have a limit point.**

The necessity is well known. It remains only to establish the sufficiency.

PROOF. Suppose that G is a collection of domains covering a point set K. Let α denote a well-ordering of K. For each n, let H_n denote a subcollection of G obtained by the following method. Let P_1 denote the first element of α such that some element g_1 of G contains every region of G_n of Axiom 1 that contains P_1. Let P_2 denote the first element of α, not contained in g_1, such that some element g_2 of G contains every region of G_n of Axiom 1 that contains P_2. In general, if $\bar{z}$ is an ordinal and for each ordinal z, $z<\bar{z}$, P_z and g_z are chosen, then let $P_{\bar{z}}$ denote the first point (if any) in α not contained in $\sum g_z$, $z<\bar{z}$, such that some element $g_{\bar{z}}$ of G contains every region of G_n of Axiom 1 that contains $P_{\bar{z}}$. From this construction, it is clear that the set $P_1, P_2, P_3, \cdots, P_z, \cdots$ has no limit point, for no region of G_n contains more than one of them. Hence $H_n=g_1, g_2, g_3, \cdots, g_z, \cdots$ is a countable subcollection of G. Then $G'=\sum_1^\infty H_n$ is a countable subcollection of G. Furthermore, G' covers K. For suppose that there is a point P of K not contained in any element of G'. Let g denote a domain of G containing P. By

* Lemma B is an advance over Theorem 18 on page 14 of R. L. Moore's *Foundations of Point Set Theory*. However, Moore's arguments may be used with some modifications to establish Lemma B.

Axiom 1 there exists a number n such that every region of G_n which contains P lies in g. Hence P, for some ordinal $\bar{z}$, is $P_{\bar{z}}$ used in the selection of H_n, which is a contradiction.

LEMMA C. *If every uncountable subset of a Moore space M has a limit point, M is a completely separable metric space.*

PROOF. By Lemma B, for each n, G_n of Axiom 1 contains a countable subcollection G_n' covering M; hence $G'=\sum G_n'$ is a countable collection of regions having the property that if P is a point of a region R, some element of G' contains P and lies in R. Hence M is completely separable. Professor Moore has pointed out that such a space is metric.*

THEOREM 5. *If $2^{\aleph_1}>2^{\aleph_0}$, then every separable normal Moore space M is completely separable and metric.*

This follows from Theorem 4 and Lemma C.

The author has tried for some time without success to prove that $2^{\aleph_1}>2^{\aleph_0}$. But although Theorem 5 is unsatisfactory in this respect, it does raise a question of some interest: *Is every normal Moore space M metric?* This question is as yet unsettled. However, if the answer is *yes*, then it should be possible to establish directly certain results for normal Moore spaces M which are known to hold in metric spaces but which are known not to hold in all Moore spaces M. The author has established a number of such theorems but it seems likely that only one of them may be of use in settling the question itself.

THEOREM 6. *A normal Moore space M is completely normal.*

PROOF. Suppose that H and K are two mutually separate subsets of a normal Moore space M. For each integer n, let H_n denote the set of all points P of $\overline{H}$ such that no region of G_n of Axiom 1 which contains P contains a point of $\overline{K}$.† Likewise, for each n, let K_n denote the set of all points P of $\overline{K}$ such that no region of G_n which contains P contains a point of $\overline{H}$. For each n, H_n and K_n are closed and $H \subset \sum H_n$ and $K \subset \sum K_n$. Let D_{H_1} denote a domain containing H_1 such that $\overline{D}_{H_1} \cdot \overline{K}=0$. Let D_{K_1} denote a domain containing K_1 such that $\overline{D}_{K_1} \cdot \overline{(H+D_{H_1})}=0$.

* R. L. Moore, *Foundations of Point Set Theory*, pp. 459 and 464.

† The notation $\overline{K}$ means K plus its limit points.

Let D_{H_2} denote a domain containing H_2 such that $\overline{D}_{H_2} \cdot \overline{(K+D_{K_1})} = 0$. Let D_{K_2} denote a domain containing K_2 and such that $\overline{D}_{K_2} \cdot \overline{(H+D_{H_1}+D_{H_2})} = 0$. This process may be continued and $D_H = \sum D_{H_n}$ and $D_K = \sum D_{K_n}$ are two mutually exclusive domains covering H and K respectively.

THE UNIVERSITY OF TEXAS

ON AN INTEGRAL EQUATION WITH AN ALMOST PERIODIC SOLUTION

BY B. LEWITAN

We assume that the function $f(x)$ is almost periodic in the sense of H. Bohr and that the functions $E(\alpha)$, $\alpha E(\alpha)$ are absolutely integrable in $[-\infty, \infty]$.

THEOREM. *If all real zeros of the function*

$$\gamma(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E(u)e^{-i\alpha u}du$$

have integer multiplicities and only two limit points ∞, α^*, *then every solution* $\phi(x)$ *of the equation*

$$\int_{-\infty}^{\infty} E(\xi - x)\cdot\phi(\xi)d\xi = f(x) \tag{1}$$

which is uniformly continuous and bounded in $[-\infty, \infty]$ *is almost periodic.*

PROOF. Without loss of generality we may assume that the finite limit point α^* has the value 0; otherwise we multiply equation (1) by $e^{-i\alpha^* x}$.

Putting

$$f_n(x) = \frac{3}{2\pi}\int_{-\infty}^{\infty} f\left(x + \frac{2u}{n}\right)\frac{\sin^4 u}{u^4}\,du,$$

we obtain

$$\int_{-\infty}^{\infty} E(\xi)\phi_n(\xi + x)d\xi = f_n(x),$$

where

$$\phi_n(t) = \frac{3}{2\pi}\int_{-\infty}^{\infty} \phi\left(t + \frac{2u}{n}\right)\frac{\sin^4 u}{u^4}\,du.$$

If $v_n(\alpha)$ denotes the generalized Fourier transform of $\phi_n(t)$, then, in our case,† $v_n(\alpha)$ is a linear function for $\alpha > 2n$ and $\alpha < -2n$.

The functions $f_n(x)$ and $\phi_n(x)$ are differentiable and the derivative of $\phi_n(x)$ is bounded. The function $E(\xi)$ being absolutely integrable, we therefore obtain

$$\int_{-\infty}^{\infty} E(\xi)\phi_n'(\xi + x)d\xi = f_n'(x).$$

Putting

$$\lambda_\epsilon(\alpha) = \begin{cases} 1 & \text{for } |\alpha| \leqq \epsilon, \\ \left(2 - \dfrac{|\alpha|}{\epsilon}\right)^2\left(2\dfrac{|\alpha|}{\epsilon} - 1\right) & \text{for } \epsilon \leqq |\alpha| \leqq 2\epsilon, \\ 0 & \text{for } |\alpha| \geqq 2\epsilon, \end{cases}$$

and

$$\tau_\epsilon(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lambda_\epsilon(\alpha)e^{-i\alpha u}d\alpha,$$

$$f_{n,\epsilon}(x) = f_n'(x) - \int_{-\infty}^{\infty} f_n'(x + u)\tau_\epsilon(u)du,$$

$$\phi_{n,\epsilon}(x) = \phi_n'(x) - \int_{-\infty}^{\infty} \phi_n'(x + u)\tau_\epsilon(u)du,$$

we obviously have

$$\int_{-\infty}^{\infty} E(\xi)\cdot\phi_{n,\epsilon}(\xi + x)d\xi = f_{n,\epsilon}(x).$$

If $v_{n,\epsilon}(\alpha)$ and $u_{n,\epsilon}(\alpha)$ are generalized Fourier transforms of $\phi_{n,\epsilon}(x)$ and $f_{n,\epsilon}(x)$, then the relation‡

† S. Bochner, Mathematische Annalen, vol. 102 (1929), pp. 489–504, vol. 103 (1930), pp. 588–597.

‡ S. Bochner, loc. cit.

$$\gamma(\alpha)dv_{n,\epsilon}(\alpha) = du_{n,\epsilon}(\alpha)$$

holds.

It follows from the construction of the function $\lambda_\epsilon(\alpha)$ that the function $\gamma(\alpha)$ has a finite number of zeros in those intervals where $v_{n,\epsilon}(\alpha)$ is not linear. Consequently, by a result of S. Bochner,† the function $\phi_{n,\epsilon}(x)$ is almost periodic in the sense of H. Bohr.

When $\epsilon \to 0$, $\phi_{n,\epsilon}(x)$ converges to $\phi_n'(x)$ uniformly in $[-\infty, \infty]$. This follows from

$$\int_{-\infty}^{\infty} \phi_n'(x+u)\tau_\epsilon(u)du = \epsilon \int_{-\infty}^{\infty} \phi_n(x+u)\frac{\tau_\epsilon'(u)}{\epsilon}du \leqq \epsilon M \to 0,$$

where M is a constant. Hence, $\phi_n'(x)$ is almost periodic in the sense of H. Bohr. But $\phi_n(x)$ is bounded. Therefore, by the theorem of Bohr, $\phi_n(x)$ is also almost periodic. Finally, $\phi(x)$ being uniformly continuous, the sequence $\phi_n(x)$ converges to $\phi(x)$ uniformly in $[-\infty, \infty]$ as $n \to \infty$, and $\phi(x)$ is almost periodic itself.

We note that the assertion of the theorem remains valid if, more generally, the limit points of the zeros of $\gamma(\alpha)$ are isolated; it is also possible to drop the assumption that the zeros have integer multiplicities.

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† S. Bochner, loc. cit.

AN INDECOMPOSABLE LIMIT SUM

BY N. E. RUTT

It is the object of this paper to investigate a certain simple monotone sequence of continua. The theorem of the paper states conditions under which the limit sum of the sequence is indecomposable. The precise formulation and proof of the theorem will be undertaken after the following lemma is established.

LEMMA. *Let K be a plane bounded indecomposable continuum and L a plane bounded continuum such that $K \cdot L \neq 0$, and that $c(L)$* includes a particular component λ containing the component δ of $c(L+K)$ with the following properties:*

(a) *the set L contains two distinct points, a and c, connected through δ by the arc B which divides δ into δ_i and δ_e, and λ into λ_i and λ_e;*

(b) *both λ_i and λ_e contain points of K.*

Then each component of $c(K+L)$ has as its boundary a proper subset of $K+L$.

The assumption that $c(K+L)$ has a component γ with boundary Γ such that $\Gamma \supset (K+L)$ will be shown contradictory. Let the boundaries of δ_i, δ_e, λ_i, λ_e be respectively Δ_i, Δ_e, Λ_i, and Λ_e. Suppose that δ is unbounded and also δ_e and λ_e, so that δ_i and λ_i will necessarily be bounded. Evidently $\lambda_i \supset \delta_i$ and $\lambda_e \supset \delta_e$. Consider first the case in which L is irreducible between a and c.

Both Λ_i and Λ_e contain L. For $\Lambda_i \subset L+B$ and $\Lambda_e \subset L+B$; so, since B is an arc with $L \cdot (B)$† $=0$, $\Lambda_i \cdot L$ and $\Lambda_e \cdot L$ are continua containing $a+c$. If either of these is not identical with L, then L is reducible between a and c. The domains δ and γ are, moreover, identical, for both λ_i and λ_e contain points of K, therefore points of Γ, and therefore points of γ. There is thus an arc X in γ such that $X \cdot \lambda_i \neq 0$ and $X \cdot \lambda_e \neq 0$, and since $X \cdot L = 0$, then $(B) \cdot X \neq 0$. This implies $X \cdot \delta \neq 0$, accordingly $\gamma \cdot \delta \neq 0$; and as both γ and δ are components of $c(K+L)$, then $\gamma = \delta$, and $\lambda + L \supset \Gamma \supset K$.

Let K_i be the sum of $K \cdot \lambda_i$ and of all the components of $L \cdot K$

* If X is a point set then $c(X)$ is the complement of X.

† If X is an arc then (X) is X with ends omitted.

containing limit points of $K\cdot\lambda_i$; and let K_e be a similar set constructed from λ_e. Each of these is a closed subset of K; and $K_i+K_e \supset K\cdot(\lambda_i+\lambda_e) \supset K\cdot\lambda$, so that $K_i+K_e \supset \overline{K\cdot\lambda}$. If $K_i+K_e \not\supset K\cdot L$, then $K\cdot L$ includes a component M such that $\overline{K\cdot\lambda}\cdot M=0$ and M may be enclosed in a simple closed curve C, not intersecting $K\cdot L$, and excluding $\overline{K\cdot\lambda}$. But $C\cdot(\overline{K\cdot\lambda}+K\cdot L)=0$ implies $C\cdot K=0$, because $\lambda+L \supset K$, so C separates K without intersecting it. Accordingly $K_i+K_e=K\cdot\lambda+K\cdot L=K$. But now if both K_i and K_e were continua, as they are proper subsets of K, the set $K=K_i+K_e$ would be decomposable. Thus either K_i or K_e is disconnected.

Let K_i be disconnected, that is, let $K_i=K_a+K_c$ where $K_a=\overline{K_a}$, $K_c=\overline{K_c}$, and $K_a\cdot K_c=0$. As $\lambda_i\cdot K$ is contained by Δ_i, the components of $K\cdot\lambda_i$ are an orderable collection and elements of this collection accessible from δ_i must belong to both K_a and K_c. Thus there is a pair of arcs B_a and B_c where $\delta_i \supset \{(B_a)+(B_c)\}$, $B_a\cdot K_a\cdot\lambda_i\neq 0$, $B_c\cdot K_c\cdot\lambda_i\neq 0$, $B_a\cdot(B)\neq 0$, $B_c\cdot(B)\neq 0$, and $B_a\cdot B_c=0$. Now B_a+B+B_c contains an arc D, such that $D \supset B_a+B_c$, $D\cdot B$ is a subarc of (B), $(D) \subset \delta$, and $\delta-(D)$ is a pair of domains δ_u and δ_b. Let the notation be so chosen that $\delta_b \subset \delta_i$ and $\delta_u \supset \delta_e$, the relationships being clear from the construction; and let the boundaries of δ_u and δ_b be Δ_u and Δ_b. Note that $\delta_b\cdot(a+c)=0$.

Now $\Delta_b\cdot(K+L) \not\subset K$. For $\Delta_b \supset D$ and consequently $\Delta_b\cdot K_a\neq 0$ and $\Delta_b\cdot K_c\neq 0$. But $\Delta_b\cdot(K+L)$ is a continuum because $\Delta_b\cdot c(K+L)=(D)$. As $K_a\cdot K_c=0$, $\overline{\Delta_b\cdot K_a}\cdot\overline{\Delta_b\cdot K_c}=0$; that is $\Delta_b\cdot(K+L) \not\subset \Delta_b\cdot(K_a+K_c)$. Therefore $\Delta_b\cdot L\cdot c(K_a+K_c)=L_k\neq 0$. Moreover L_k includes a component L_b such that $\overline{L_b}\cdot K_a\neq 0$ and $\overline{L_b}\cdot K_c\neq 0$, for otherwise it must follow again that $\Delta_b-(D)$ is disconnected. Now $K_e \not\supset L_b$, as otherwise there must be a component of $K\cdot L$ containing L_b and thus contained in both K_a and K_c. Thus $K_i+K_e \not\supset L_b$ and $L\cdot c(K) \supset h$, a point distinct from a and from c, and in Δ_b.

Let S be a circle with center h and radius such that $ce(S)$* is a set with no points in $K+D+B$, and such that in $e(S)\cdot\delta_e$ there is a ray R with end on (B). As L is irreducible between a and c, $L\cdot ci(S)$ consists of sets L_a'' and L_c'' mutually separated

* If S is a simple closed curve, its interior is $i(S)$ and its exterior is $e(S)$. Accordingly $ci(S)=S+e(S)$ and $ce(S)=S+i(S)$.

between a and c. Let L_a and L_c be the components of these containing a and c. Evidently $S \cdot L_a \neq 0$ and $S \cdot L_c \neq 0$. Let F be an arc such that $F \cdot S \neq 0$, $(F) \subset e(S)$, $(F) \subset \delta_b$, and $F \cdot (D \cdot B) \neq 0$. Let E be an arc such that $E \cdot S \neq 0$, $(E) \subset e(S)$, $(E)+E \cdot S \subset \lambda_e$, and $E \cdot B = F \cdot B$. Let G be an arc such that $(G) \subset i(S)$ and $G \cdot S = (E+F) \cdot S$. The set $E+F+G$ is a simple closed curve J, intersecting L only in $i(S)$.

The set $L_a+ce(S)+F+B$ bounds a bounded domain ϕ_a containing points of $K \cdot c(L) \cdot \lambda_i$; and $L_c+ce(S)+F+B$ bounds a similar domain ϕ_c. For $L_a+ce(S)$ is obviously a continuum and $B+F$ contains a cut F_a of its complement. Of the two components of $c(F_a+L_a+ce(S))$ which are bounded in part by (F_a) let ϕ_a be the bounded one. The boundary of ϕ_a contains either $B \cdot B_a$ or $B \cdot B_c$ (suppose the former), but not of course both. Thus by elementary reasoning from the constructions used, it appears that $\phi_a \supset B_a \cdot c(B)$ and thus contains a component of $K_a \cdot \lambda_i$. In a similar way ϕ_c may be proved to contain a component of $K_c \cdot \lambda_i$. Moreover $\phi_a \cdot \phi_c = 0$ and if $i(J) \supset \phi_a$ then $e(J) \supset \phi_c$ and vice versa. To be explicit, assume that $i(J) \supset \phi_a$.

That the domain λ_e contains an uncountably infinite number of components of $\lambda_e \cdot K$ each of which has a disconnected set of limit points in H will now be shown. Since $i(J) \supset \phi_a$ and $e(J) \supset \phi_c$, both $i(J)$ and $e(J)$ contain points of $K \cdot c(H)$ and thus contain points of every composant of K. Let $[Q_\alpha]$ be a collection of subcontinua of K, one and only one in each composant of K, and each one having both a point in ϕ_a and a point in ϕ_c. The elements of $[Q_\alpha]$ are uncountable and mutually exclusive. Any one, Q_g, of $[Q_\alpha]$ has a point in Φ_a and one in Φ_c, where Φ_a and Φ_c are the boundaries respectively of ϕ_a and ϕ_c; so $Q_g \cdot \Phi_a = Q_g \cdot (L_a+S+F_a) = Q_g \cdot L_a \neq 0$, and also $Q_g \cdot L_c \neq 0$. It appears indeed that Q_g has a point in each of the mutually separated closed sets $L \cdot e(J) \cdot ci(S)$ and $L \cdot i(J) \cdot ci(S)$, and as these two sets contain $L \cdot K$ they contain $L \cdot Q_g$. In consequence $Q_g \cdot c(L)$ has a component G_g with a limit point in $L \cdot i(J)$ and a limit point in $L \cdot e(J)$. As $L \cdot i(J)$ and $L \cdot e(J)$ are mutually separated, the limit set of G_g is disconnected since it includes no point of J; and also $G_g \cdot J = G_g \cdot (E+F+G) = G_g \cdot E \neq 0$, so $G_g \cdot \lambda_e \neq 0$ and therefore $\lambda_e \supset G_g$. Regard now the components $[K_\alpha]$ of $\lambda_e \cdot K$ which contain the members of $[G_\alpha]$. Each of these has limit points in both $i(J)$ and $e(J)$ and none in J, and so

has a disconnected limit set in L. No two of these are identical, for no one, such as K_g, of them has $\overline{K_g - K} = K$ as it contains none of the points of K in $\phi_a + \phi_c$, and if K_g were to contain two of $[G_\alpha]$ then K_g would be a proper subcontinuum of K containing points of two different composants of K. Thus the collection $[K_\alpha]$ is one of the sort required.

Each set $K_g + L$ is therefore a subcontinuum of $K+L$ separating the plane, a bounded component of its complement being δ_g. No pair of elements, δ_p and δ_q, of $[\delta_\alpha]$ can have a point in common unless one contains the other, for $K_p \cdot K_q = 0$. Moreover if $\delta_p \supset \delta_q$ then $\delta_p \supset K_q$, and so $\delta_p \supset \gamma$, a contradiction as $\gamma \supset \delta$ was unbounded. Thus $[\delta_\alpha]$ is an uncountable collection of mutually exclusive domains in the plane, another contradiction establishing at last the lemma for this case.

None of the undiscussed suppositions made above requires any more justification than a suitable inversion of the plane except the assumption that L is irreducible between a and c. But if L is not irreducible between a and c, then it contains a subcontinuum W which is irreducible between a and c. By examining K and W it may be seen that the hypotheses of the lemma are fulfilled, so that the set $c(K+W)$ has no component with boundary $K+W$. Neither then does the less inclusive set $c(K+H)$.

THEOREM. *If $[D_i]$ is a simple infinite sequence of plane point sets such that, for each positive integer i, D_i is indecomposable and $D_i \subset D_{i+1}$, and such that the set $\overline{\sum_1^\infty D_i}$ is a plane bounded continuum Γ which is the frontier of γ, a component of its complement, then Γ is also indecomposable.*

The theorem is obvious if no more than a finite number of $[D_i]$ are distinct, as then Γ is identical with one of $[D_i]$. Assume accordingly that all of $[D_i]$ are different, other possible cases being not significant. Let $\sum_1^\infty D_i = D_m$ and $\Gamma - D_m = D_n$. Every point of D_n is a limit point of D_m, for $\Gamma = \overline{\sum_1^\infty D_i} = \overline{D_m}$ so $\overline{D_m} \supset D_n$. Every subcontinuum of D_n is a continuum of condensation of Γ, because when D_n contains the continuum K, then $\overline{D_m} \supset D_n \supset K$ implies $\overline{\Gamma - K} \supset \overline{\Gamma - D_n} \supset \overline{D_m} \supset K$. Moreover let d be a point of D_m. Now $D_{j+1} \supset D_j$ while $D_{j+1} - D_j \neq 0$, so D_j belongs to a single composant of D_{j+1}. Thus $\overline{D_{j+1} - D_j} \supset D_j$; that

is $\overline{\Gamma-D_j} \supset \overline{D_{j+1}-D_j} \supset D_j$, or D_j is a closed set nowhere dense in the closed set Γ. But this being true for any value of j $(j=1, 2, 3, \cdots)$ then $\sum_1^\infty D_j = D_m$ must be a set of the first category in the closed set Γ. Thus D_n is a set of the second category in Γ everywhere dense in the set Γ. That is, $\overline{D_n} \supset D_m$ so $\overline{D_n} \supset d$. Consequently every subcontinuum of D_m is a continuum of condensation of Γ, for when $D_m \supset K$, then $\overline{D_n} = \Gamma$ implies that $\overline{\Gamma-K} \supset \overline{\Gamma-D_m} \supset \overline{D_n} \supset K$. Henceforth consider γ unbounded.

The argument will be completed by showing that every proper subcontinuum of Γ is a continuum of condensation of Γ. Let K be such a continuum and, as the cases $D_m \supset K$ and $D_n \supset K$ have already been dealt with, suppose that $K \cdot D_m \neq 0$ and $K \cdot D_n \neq 0$. Clearly $K \not\supset D_m$ for $K \supset D_m$ would imply $K = \Gamma$. There must be some element of $[D_i]$ contained in part but not entirely by K. Let D_k be such an element. Thus, if $i > k$, the element D_i can not be a subset of K, for $K \supset D_i$ implies $K \supset D_i \supset D_k$.

Suppose that the set $\overline{D_k \cdot c(K)} = \Gamma_k$ is not connected. As $D_k + K \subset \Gamma$, $c(D_k+K) \supset c(\Gamma) \supset \gamma$. Thus there is a connected domain γ_k complementary to D_k+K such that $\gamma_k \supset \gamma$. As $\bar{\gamma}_k \supset \bar{\gamma} \supset \Gamma$ then $\bar{\gamma}_k \supset D_k+K$. But $\bar{\gamma}_k = \gamma_k + B_k$, where B_k is the boundary of γ_k, and so $\bar{\gamma}_k \supset D_k+K$ implies $B_k \supset D_k+K$. Let G be a component of Γ_k containing the end of a ray R_g contained except for its end in γ_k. As every component of Γ_k consists of limit points of γ_k, and Γ_k is not connected, there is another component H of Γ_k containing the end of another ray R_h, $(R_g \cdot R_h = 0)$, which is except for its end contained in γ_k.

Now there exists a simple closed curve C such that $i(C) \supset G$, $C \cdot \Gamma_k = 0$, $e(C) \supset H + R_h$, and $C \cdot R_g$ is a single point. Upon tracing C in opposite directions from $C \cdot R_g$, first points of D_k+K are clearly encountered. Let the subarc of C thus identified be B. But $(B) \cdot (D_k+K) = 0$ by selection, and $B \cdot (D_k+K) \subset B \cdot (\Gamma_k+K) \subset C \cdot \Gamma_k + B \cdot K \subset B \cdot K \subset K$, so B is a cut of the unbounded complementary domain γ_d of K. Thus $\gamma_d - (B)$ consists of two domains, a bounded one γ_b and an unbounded one γ_u. As $R_h \cdot (B+K) \subset R_h \cdot (C+K) \subset R_h \cdot C + R_h \cdot K = 0$, then $\gamma_u \supset R_h$, and thus $\gamma_u \cdot D_k \neq 0$, for indeed $\gamma_u \supset H$. Upon considering γ_b it may be seen with reasonable ease that the single point $B \cdot R_g$ separates R_g into two parts, the unbounded one of which is a subset of γ_u whereas the bounded one is a subset of γ_b. As the end of

the bounded part is in G, then γ_b also contains points of Γ_k as it contains G. These facts make it clear also that the ends of B are distinct, for if identical they would coincide with a cut point of the continuum D_k, although D_k is indecomposable.

A contradiction of the lemma now appears, for this is the situation: the plane continuum D_k is indecomposable and bounded and K is a bounded continuum such that $K \cdot D_k \neq 0$, the set K contains two distinct points connected in $c(D_k+K)$ by an arc B having only its ends in common with D_k+K and separating the component of $c(K)$ which contains it into two domains γ_b and γ_u, both γ_b and γ_u contain points of D_k, and there is a component of $c(D_k+K)$ whose boundary is identical with D_k+K. As this is ridiculous, the set Γ_k is connected as was to be proved.

But if K is not a continuum of condensation of Γ, there exists a point s of K and a circle S such that $i(S) \supset s$ and $e(S) \supset \overline{\Gamma - K}$. But $\overline{D_m} = \Gamma$ so $i(S) \cdot D_m \neq 0$; that is, there exists a subscript q such that $i(S) \cdot D_q \neq 0$. For any subscript $j > q$, then $i(S) \cdot D_j \neq 0$ as $i(S) \cdot D_j \supset i(S) \cdot D_q \neq 0$. Let r be a natural number greater than k and greater than q. Then $\overline{D_r \cdot c(K)} = \Gamma_r$ is non-vacuous and connected as has been seen already. But $\Gamma_r \cdot i(S) \subset \overline{\Gamma - K} \cdot i(S) = 0$, so Γ_r fails to contain any point of the non-vacuous set $i(S) \cdot D_r$. Therefore Γ_r is a proper subcontinuum of D_r, and must accordingly belong to a single composant D_r^a of D_r. As $D_r \cdot c(K) \neq 0$ by supposition, there are points of D_r not in K. But let D_r^b be a second composant of D_r. As $D_r^a \supset \Gamma_r$, then $D_r^b \subset K$. Accordingly $\overline{D_r^b} \subset K$, and as $\overline{D_r^b} = D_r$, finally $D_r \subset K$, a contradiction.

As the contradiction is now general, the theorem is proved.

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CYCLIC RELATIONS IN POINT SET THEORY*

BY E. C. STOPHER, JR.

1. *Introduction.* The formula

$$\phi c\phi c\phi c\phi A = \phi c\phi A, \tag{1}$$

where c denotes the operation of taking complements and ϕ is an arbitrary operator, is of considerable interest in the study of sets of points. Kuratowski† proved that the formula holds for a postulated closure or extension function. Zarycki‡ established the formula in case ϕA is the "interior" of A and Sanders§ established it for a general derived set operator satisfying the postulates:

I. $\quad d(A + B) = dA + dB,$

II. $\quad d^2A \leqq dA.$

In this paper we shall follow the established‖ practice of postulating a derived set operator d, subject to I and II, and, using a notation introduced by Chittenden, define certain terms as follows:

Identity: $\quad 1A = A;$

Complement: $\quad cA = S - A$ (S denotes the entire space);

Extension:¶ $\quad eA = A + dA;$

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* Presented to the Society, April 10, 1937.

† C. Kuratowski, *Sur l'opération A de l'analysis situs*, Fundamenta Mathematicae, vol. 3 (1922), pp. 182–199.

‡ M. Zarycki, *Notions fondamentales de l'analysis situs*, Fundamenta Mathematicae, vol. 9 (1927), pp. 3–15.

§ S. T. Sanders, Jr., *Derived sets and their complements*, this Bulletin, vol. 42 (1936), pp. 577–584.

‖ F. Riesz, *Stetigkeitsbegriff und abstrakte Mengenlehre*, Atti del 4 Congresso Internationale dei Matematici, Roma, 1910, vol. 2, p. 18; Chittenden, *On general topology and the relation of the properties of the class of all continuous functions to the properties of space*, Transactions of this Society, vol. 31 (1929), pp. 290–321.

¶ F. Hausdorff, *Mengenlehre*, pp. 109–129. Under the postulates given, the derived set corresponds to Hausdorff's set of β points, A_β. The extension corresponds to his α points, A_α. Similarly, hA corresponds to A_h, jA to A_j, bA to A_r (border is a translation of the German word "rand"), kA to A_k, and sA to A_s.

Interior:	$iA = AcdcA$;
Concentrated part:	$hA = AdA$;
Isolated part:	$jA = AcdA$;
Border:	$bA = AdcA$;
Frontier:	$fA = AdcA + cAdA$;
Kernel:	$kA = \sum B \leqq A$, such that $B \leqq dB$;
Separated part:	$sA = AckA$.

In §§2–7 it will be shown that each of these operators except b and h satisfies the Kuratowski formula. It might be pointed out that these operators do not all have the same basic properties. In contrast to Postulates I and II we have, for example,

$$i(A + B) \geqq iA + iB,$$
$$i(AB) = iA\,iB,$$
$$i^2A = iA,$$
$$f(A + B) \leqq fA + fB,$$
$$f^2A \leqq fA.$$

This list of operators is by no means a complete list of operators satisfying the Kuratowski formula, as can be readily seen by considering the formula

$$\psi^4 A = \psi^2 A. \tag{2}$$

We have the proposition that if an operator θ satisfies formula (1) ((2)), its transform* satisfies (1) ((2)) and its complement satisfies (2) ((1)). We make use of this proposition in §8 to obtain additional operators satisfying the Kuratowski formula.

Examples will be given in §9 to show that the b and h operators do not in general satisfy the Kuratowski formula. It will be shown in §§10 and 11 that each of $(bc)^\beta bA$ and $(hc)^\beta hA$ with increasing β defines a set.

2. *Identity, Complement.* The identity operator can be used in the Kuratowski formula since, on account of the relation $c^2A = A$, each side of the equation reduces to cA. It is readily seen also that the complementary operator can be used in place of ϕ, each side of the equation again reducing to cA.

* An operator ϕ is said to be the transform of an operator ψ, if $\phi A = c\psi cA$; it is the complement of the operator ψ, if $\phi A = c\psi A$.

3. *Extension, Interior.* The general "defined" extension function satisfies the Kuratowski formula since this function satisfies Kuratowski's postulates I, II, and IV, which Kuratowski* showed were sufficient. We immediately have

$$icicicA = iciA,$$

since the interior and extension operators are transforms of each other.

4. *Frontier.* Since $fcA = fA$, $f^2A = bfA$, and $b^2A = bA$, both $fcfcfcfA$ and $fcfA$ reduce to bfA, establishing the formula

$$fcfcfcfA = fcfA.$$

5. *Isolated Part.* By definition,

$$\begin{aligned} jA &= AcdA, \\ cjA &= cA + hA, \\ jcjA &= (cA + hA)cdcAcdhA \\ &= cAcdcAcdhA(cdjA + djA) + hAcdhAcdcA \\ &= cAcdcAcdhAcdjA + cAcdcAcdhAdjA \\ &\quad + AdAcdhAcdcA \\ &= cAcd(cA + hA + jA) + cAcdcAcdhAdjA \\ &\quad + AdjAcdhAcdcA \\ &= cAcdS + djAcdhAcdcA(cA + A) \\ &= JcA + jdScdcA. \dagger \end{aligned}$$

Replacing A by cjA, we have

$$\begin{aligned} jcjcjA &= JjA + jdScdjA \\ &= JA + jdScdA, \end{aligned}$$

since $cdA = cdjAcdhA$ and $cdhA$ includes jdS. Again replacing A by cjA, we obtain

$$\begin{aligned} jcjcjcjA &= JcjA + jdScdcjA \\ &= JcA + JhA + jdScdcAcdhA \\ &= JcA + jdScdcA, \end{aligned}$$

* C. Kuratowski, loc. cit.

† $J = cdS$. Symbol used by Sanders, loc. cit.

again because $cdhA$ includes cd^2S, which in turn includes jdS. This establishes the formula

$$jcjcjcjA = jcjA.$$

6. *Separated Part.* By definition,

$$sA = AckA,$$
$$csA = cA + kA,$$
$$scsA = (cA + kA)ckcsA.$$

To evaluate the set $kcsA$ we make use of the fact that for any set B, kB is equal to the limit as β increases of $h^\beta B$. By definition,

$$hcsA = (cA + kA)(dcA + dkA)$$
$$= hcA + dkA,$$

since the product of sA and dkA is null; and

$$h^2csA = (hcA + dkA)(dhcA + dkA)$$
$$= h^2cA + dkA,$$

dkA being perfect, that is, $d^2kA = dkA$. Continuing, we see that

$$h^\beta csA = h^\beta cA + dkA,$$
$$kcsA = kcA + dkA = csAkS.$$

Hence,

$$ckcsA = sA + sS,$$
$$scsA = (cA + kA)(sA + sS)^\cdot$$
$$= cAsS,$$

kA being a subset of kS and not of sS.

Substituting csA for A, we have

$$scscsA = sAsS = AsS.$$

Again substituting csA for A, we obtain

$$scscscsA = csAsS = cAsS + kAsS$$
$$= cAsS,$$

establishing the formula

$$scscscsA = scsA.$$

7. *Kernel.* By definition,

$$\begin{aligned} kA &= AcsA, \\ ckA &= cA + sA, \\ kckA &= kcA + sAdkcA. \end{aligned}$$

Substituting ckA for A, we have

$$\begin{aligned} kckckA &= k^2A + sckAdk^2A \\ &= kA + sckAdkA. \end{aligned}$$

We observe that

$$\begin{aligned} sckA &= ckAckckA \\ &= (cA + sA)ckcA(csA + cdkcA) \\ &= scA + sAcdkcA \\ &= scA + AsS, \end{aligned}$$

since $sAcdkcAkS=0$. Therefore,

$$\begin{aligned} kckckA &= kA + (scA + AsS)dkA \\ &= kA + scAdkA. \end{aligned}$$

Again substituting ckA for A, we have

$$\begin{aligned} kckckckA &= kckA + skAdkckA \\ &= kckA, \end{aligned}$$

since $skA=0$.

8. *Additional Operators.* Since the transforms of operators satisfying the Kuratowski formula also satisfy it, we immediately obtain the fact that the following operators satisfy the Kuratowski formula: cdc, $cjc=(1+hc)$, $cfc=(i+ic)$, $ckc=(1+sc)$, and $csc=(1+kc)$.

Since e, i, j, b, f, k, and s are known to satisfy equation (2), it follows immediately that ic, $(c+dc)$, $(c+h)$, $(c+i)$, $(i+ic)$, $(c+s)$, and $(c+k)$ satisfy the Kuratowski formula.

9. *Examples.* Equation (1) will not hold in general for ϕ equal to either of the remaining two operators, b and h, as the following examples show.

Let the space S be the closed linear interval (0, 1). Let the set A be the points $(1/2, 3/4, 7/8, \cdots)$. Then,

$$\begin{aligned} bA &= A, \\ bcbA &= 1, \\ bcbcbA &= \text{null}, \\ bcbcbcbA &= \text{null} \\ &\neq bcbA. \end{aligned}$$

For the second example, consider the same space S but include also the point 1 in the set A. Then,

$$\begin{aligned} hA &= 1, \\ hchA &= S - 1, \\ hchchA &= \text{null}, \\ hchchchA &= S \\ &\neq hchA. \end{aligned}$$

Although equation (1) does not hold for either b or h, it is interesting to note in these examples that

$$\begin{aligned} bcbcbcbcbA &= bcbcbA, \\ hchchchchA &= hchchA. \end{aligned}$$

10. *Border.* Examining further the operator b, we see that

$$cbA = cA + iA,$$

and

$$\begin{aligned} bcbA &= (cA + iA)dbA \\ &= bcAdbA, \end{aligned}$$

since dbA is included in fA rather than in iA.

Substituting cbA for A and making use of the fact that $c^2=1$ and $b^2=b$, we have

$$\begin{aligned} bcbcbA &= bccbAdbcbA \\ &= bAdbcbA \\ &\leqq bA. \end{aligned}$$

Thus we see that we have two monotonic decreasing se-

quences of sets, $(bc)^{2\beta}bA$ and $(bc)^{2\beta+1}bA$, $\beta=1, 2, 3, \cdots$. Chittenden has pointed out that, since a set is determined by every product ΠA_β, $(\beta=1, 2, 3, \cdots)$, where $[A_\beta]$ represents a monotonic decreasing sequence of sets, each of these sequences defines a set and we have implied that for all ordinals α greater than or equal to some finite or transfinite ordinal α_0,

$$(bc)^{\alpha+2}bA = (bc)^{\alpha}bA.$$

11. *Concentrated Part.* Similarly, it can be shown that, with increasing β, $(hc)^{2\beta}hA$ defines a set. We have

$$\begin{aligned} hA &= AcjA, \\ chA &= cA + jA, \\ hchA &= (cA + jA)(dcA + djA) \\ &= hcA + jAdcA + cAdjA. \end{aligned}$$

Substituting chA for A, we may write,

$$(hc)^2hA = h^2A + jchAdhA + hAdjchA.$$

However,

$$\begin{aligned} jchA &= (cA + jA)cdcAcdjA \\ &= cAcdcAcdjAcdhA + cAcdcAcdjAdhA \\ &\quad + jAcdcA, \end{aligned}$$

jA being included in $cdjA$. This simplifies to

$$\begin{aligned} jchA &= cAcd(cA + jA + hA) + jcAcdjcAcdjAdhA \\ &\quad + AcdAcdcA \\ &= JcA + JA + jcAidS \\ &= J + jcAidS, \end{aligned}$$

and we have

$$\begin{aligned} (hc)^2hA &= h^2A + (J + jcAidS)dhA + hA(dJ + d\overline{jcAidS}) \\ &= kA + sh^2A + jcAidS + shAdJ \\ &= (kA + jcAidS)dkA + sh^2AidS + shAdJ, \end{aligned}$$

since $h^2A = kh^2A + sh^2A$ and $khA = kA$.

Again substituting $chchA$ for A, we have

$$\begin{aligned} (hc)^4hA = (kchchA + jhchAidS)dkchchA \\ + sh^2chchAidS + shchchAdJ. \end{aligned}$$

But,

$$\begin{aligned}
kchchA &= khchchA = kA + jcA\,idS,\\
dkchchA &= dkA + d(jcA\,idS) = dkA,\\
jhchA\,idS &= idS(hcA + jAdcA\\
&\quad + cAdjA)cdhcAcd(jAdcA)cd(cAdjA)\\
&= idS[jhcAcd(cAdjA)cd(jAdcA)\\
&\quad + j(cAdjA)cdhcAcd(jAdcA)\\
&\quad + j(jAdcA)cdhcAcd(cAdjA)]\\
&= idS\cdot jhcA,\\
sh^2chchA\,idS &= h(sh^2A\,idS) = sh^3A\,idS,\\
shchchAdJ &= shAdJ.
\end{aligned}$$

Making these substitutions, we obtain

$$\begin{aligned}
(hc)^4hA = {} & (kA + jcA\,idS + jhcA\,idS)dkA + sh^2A\,idS\\
& + shAdJ.
\end{aligned}$$

Continuing, we have

$$\begin{aligned}
(hc)^{2\beta}hA = {} & \left[kA + \sum_{r=0}^{\beta-1} jh^rcA\,idS\right]dkA\\
& + sh^{\beta+1}A\,idS + shAdJ.
\end{aligned}$$

Since

$$\begin{aligned}
scA &= jcA + jhcA + jh^2cA\cdots\\
&= \sum_{r=0} jh^rcA,
\end{aligned}$$

and $sh^\beta cA = h^\beta scA \to 0$, with increasing β,* it is apparent that with increasing β,

$$\begin{aligned}
(hc)^{2\beta}hA &\to (kA + scA\,idS)dkA + shAdJ\\
&= (kA + scA)idS + hAdJ,
\end{aligned}$$

and this establishes the proof.

---

* $\to$ is the ordinary symbol for convergence; $A_\beta \to A$, with increasing β, is equivalent to $\Pi\,(AcA_\beta + A_\beta cA) = 0$, $(\beta = 1, 2, 3, \cdots)$.

12. *Summary.* The results of §§2–7 can be summarized in the following theorem:

THEOREM. *The Kuratowski formula,*

$$\phi c\phi c\phi c\phi A = \phi c\phi A,$$

is satisfied for ϕ equal to any of the operators $1, c, d, e, i, j, f, k,$ and s.

Sections 10 and 11, together with this theorem, imply the following corollary:

COROLLARY. *The equation*

$$(\phi c)^{\alpha+2}\phi A = (\phi c)^{\alpha}\phi A$$

holds for every ordinal α equal to or greater than some finite or transfinite ordinal α_0, and for ϕ equal to any of the operators $1, c, d, e, i, h, j, b, f, k,$ and s.

UNIVERSITY OF IOWA

A CONDITION THAT A FIRST BOOLEAN FUNCTION VANISH WHENEVER A SECOND DOES NOT

BY J. C. C. MCKINSEY*

It is well known† that if two polynomials $f(x_1, \cdots, x_n)$ and $g(x_1, \cdots, x_n)$ in the field of complex numbers are such that f vanishes whenever g does not, then at least one of the two polynomials f and g is identically zero. The corresponding law, however, does not, in general, hold for Boolean functions, as may be seen by considering the two functions x and x' in a two-element Boolean algebra; the statement that either $x=0$ or else $x'=0$ in a two-element Boolean algebra is, indeed, the familiar law of excluded middle. It is the purpose of the present note to determine the conditions on the coefficients of two Boolean functions in order that the first vanish whenever the second does not.

The condition found involves *prime Boolean elements*, which are defined as follows:

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† See, for example, Bocher, *Introduction to Higher Algebra*, p. 8.

DEFINITION: *A Boolean element $p \neq 0$ is called a prime if there exists no Boolean element x such that $x \neq 0$, $x \neq p$, and $x < p$.*

The condition, for functions of one variable, is given in the following theorem. It will be noticed that to say that neither $\alpha_1 x + \alpha_2 x'$ nor $\beta_1 x + \beta_2 x'$ vanishes identically is equivalent to saying that $\alpha_1 + \alpha_2 \neq 0$ and $\beta_1 + \beta_2 \neq 0$.

THEOREM 1. *If $\alpha_1 x + \alpha_2 x'$ and $\beta_1 x + \beta_2 x'$ are two Boolean functions, neither of which vanishes identically, then a necessary and sufficient condition that, for every choice of x, either $\alpha_1 x + \alpha_2 x' = 0$ or else $\beta_1 x + \beta_2 x' = 0$ is that*

$$\alpha_1\beta_1 + \alpha_2\beta_2 = 0, \tag{1}$$

$$\alpha_1 + \beta_1 + \alpha_2 + \beta_2 = p, \text{ a prime.} \tag{2}$$

PROOF: I first show that the condition is sufficient. By (2), we have $\alpha_i = 0$ or p for $i = 1, 2$, and $\beta_i = 0$ or p, for $i = 1, 2$. By (1), we then have either $\alpha_1 = 0$ or else $\beta_1 = 0$, and either $\alpha_2 = 0$ or else $\beta_2 = 0$. Suppose $\alpha_1 = 0$, then, since $\alpha_1 + \alpha_2 \neq 0$, we have $\alpha_2 = p \neq 0$, and hence $\beta_2 = 0$, and hence, since $\beta_1 + \beta_2 \neq 0$, we have $\beta_1 = p$; thus our functions become px', and px. Suppose that, for some y, $py' \neq 0$, and $py \neq 0$. Let $z = py$; then $z = py < p$, and $z = py \neq 0$, also $z \neq p$ (since if $z = p$ we should have $0 = pp' = pz' = p(py)' = pp' + py' = py' \neq 0$). Hence p is not a prime, contrary to hypothesis; therefore, for every x, we have either $px' = 0$ or else $px = 0$. The argument is similar if we assume $\beta_1 = 0$.

I now show that the condition is necessary. Suppose that neither $\alpha_1 x + \alpha_2 x'$ nor $\beta_1 x + \beta_2 x'$ vanishes identically, and that for every x either $\alpha_1 x + \alpha_2 x' = 0$ or else $\beta_1 x + \beta_2 x' = 0$. Then for every x we have $(\alpha_1 x + \alpha_2 x')(\beta_1 x + \beta_2 x') = \alpha_1\beta_1 x + \alpha_2\beta_2 x' = 0$, and hence $\alpha_1\beta_1 + \alpha_2\beta_2 = 0$; therefore (1) holds. Moreover, the disjunction must hold, in particular, for $x = 1, 0$; hence we have either $\alpha_1 = 0$ or else $\beta_1 = 0$, and either $\alpha_2 = 0$ or else $\beta_2 = 0$. By hypothesis we cannot have $\alpha_1 = 0$ and $\alpha_2 = 0$, since this implies $\alpha_1 + \alpha_2 = 0$; and similarly we cannot have $\beta_1 = 0$ and $\beta_2 = 0$. Hence either $\alpha_1 = 0$ and $\beta_2 = 0$, or else $\alpha_2 = 0$ and $\beta_1 = 0$. If $\alpha_1 = \beta_2 = 0$, then we have, for every x, either $\alpha_2 x' = 0$ or else $\beta_1 x = 0$; hence, in particular, $\alpha_2\alpha_2'' = \alpha_2 = 0$, or $\beta_1 x_2' = 0$; or, since $\alpha_2 \neq 0$, we have $\beta_1\alpha_2' = 0$, which is equivalent to $\beta_1 < \alpha_2$; similarly $\alpha_2 < \beta_1$; it follows that $\alpha_2 = \beta_1$. Thus we have, for every x, either $\alpha_2 x = 0$ or else $\alpha_2 x' = 0$.

If α_2 is not a prime, then there is an element $c \neq 0, \alpha_2$ such that $c < \alpha_2$ (and hence such that $c\alpha_2' = 0$); then either $\alpha_2 c = 0$ or else $\alpha_2 c' = 0$; but if $\alpha_2 c = 0$, then $0 = \alpha_2 c = \alpha_2 c + \alpha_2' c = c$, contrary to hypothesis, and if $\alpha_2 c' = 0$, then $\alpha_2 c' + \alpha_2' c = 0$, so that $\alpha_2 = c$, contrary to hypothesis. Hence α_2 is a prime p, and we have $\alpha_1 + \beta_1 + \alpha_2 + \beta_2 = 0 + p + p + 0 = p$. If we suppose that $\alpha_2 = \beta_1 = 0$, the same result follows. Thus the condition is necessary.

By an induction on the number of variables in $f(x_1, \cdots, x_n)$ and $g(x_1, \cdots, x_n)$, Theorem 1 can immediately be generalized to functions of n variables, as follows:

THEOREM 2. *If $f(x_1, \cdots, x_n)$ and $g(x_1, \cdots, x_n)$ are two Boolean functions, neither of which vanishes identically, then a necessary and sufficient condition that, for every choice of $x_1, \cdots, x_n$, either $f(x_1, \cdots, x_n) = 0$ or else $g(x_1, \cdots, x_n) = 0$ is that*

$$(1) \qquad \sum_{\alpha_{i,j}=0,1} f(\alpha_{1,j}, \cdots, \alpha_{n,j}) g(\alpha_{1,j}, \cdots, \alpha_{n,j}) = 0,$$

$$(2) \qquad \sum_{\alpha_{i,j}=0,1} [f(\alpha_{1,j}, \cdots, \alpha_{n,j}) + g(\alpha_{1,j}, \cdots, \alpha_{n,j})] = p, \text{ a prime.}$$

From this theorem we see that "primeless" Boolean algebras are the only Boolean algebras that possess (vacuously) the property that, if f vanishes whenever g does not, then either f or g is identically zero. It is known that primeless Boolean algebras exist.

We may also consider the statement that a given pair f, g of Boolean functions, neither of which vanishes identically, are such that one vanishes whenever the other does not, as a generalized form of the law of excluded middle. Our theorem then asserts that to say a generalized form of the law of excluded middle holds in a Boolean algebra is equivalent to saying that the algebra is not primeless.

NEW YORK UNIVERSITY

CONCERNING SPECIAL CENTERS OF PROJECTION FOR AN ALGEBRAIC SPACE BRANCH*

BY W. S. TURPIN

1. *Introduction.* If one projects a branch of an algebraic space curve from a point (a, b, c) onto a plane and considers a, b, and c as parameters, one obtains a plane branch possessing a development whose coefficients are rational functions of a, b, and c. If it now be assumed that no relations exist between the parameters a, b, and c, this plane branch will have a certain generic composition which will change only for special values of these parameters which satisfy certain relations between the coefficients of the development. A center of projection is said to be generic with respect to a space branch and the corresponding plane projection to be a generic projection, provided the latter has a generic composition in the sense just defined.

Until recently, it had been thought that the composition of a space branch was the same as that of its generic plane projection.† However, it has been shown by example that the composition of a space branch is not necessarily the same as that of its generic projection.‡

In view of this it was deemed of interest to investigate the conditions under which a center of projection is generic with respect to a given space branch. In what follows, these conditions are determined and an explicit formulation for the locus of non-generic centers of projection is given.

2. *A Theorem on Plane Branches.* It will be convenient first to establish a theorem concerning the composition of a plane branch. The equation of such a branch, with origin at the origin of coordinates, may be written in the following manner:§

* Presented to the Society, April 11, 1936.

† See, for instance, Enriques-Chisini, *Lezioni sulla Teoria Geometrica delle Equazioni e delle Funzioni Algebriche*, vol. 2, p. 559.

‡ O. Zariski, *Algebraic Surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 3, no. 5, pp. 11–12.

§ For a detailed discussion concerning such parametric representations, see Enriques-Chisini, op. cit., vol. 2, p. 330 et seq. The notation used here is due to Zariski: O. Zariski, op. cit., p. 7.

$$
\begin{aligned}
x &= \tau^{\nu}, \\
y &= \sum_{1}^{k_1} a_{1i}\tau^{\nu_i} + b_1\tau^{m_1\nu_2} + \cdots + \sum_{1}^{k_g} a_{gi}\tau^{(m_{g-1}+i)\nu_g} + b_g\tau^{m_g} \\
&\quad + \sum_{1}^{\infty} c_i\tau^{m_g+i} \\
&= [\tau\colon a, b; m_1, n_1, \cdots, m_g, n_g],
\end{aligned} \tag{1}
$$

where

$$
\nu = n_1 n_2 \cdots n_g;\ \nu_{g+1} = 1;\ m_0 = 1;\ \nu_j = n_j n_{j+1} \cdots n_g;\ b_j \neq 0;
$$

$$
(m_j, n_j) = 1; \qquad k_j = \left[\frac{m_j - m_{j-1}n_j}{n_j}\right], \qquad (j = 1, 2, \cdots, g).
$$

Then $b_j\tau^{m_j\nu_{j+1}}$ $(j=1, 2, \cdots, g)$ are said to be the *characteristic terms* of the development of the branch, the exponents $m_j\nu_{j+1}$ are said to be the characteristic exponents of the branch, and the pairs (m_j, n_j) are said to be the characteristic pairs of the branch.

Let us now consider a branch of the form:

$$
\begin{aligned}
X &= \sum_{i=1}^{k_1} a_{1i}^{(1)} t^{\nu_i} + b_1^{(1)} t^{m_1\nu_2} + \cdots + \sum_{i=1}^{k_g} a_{gi}^{(1)} t^{(m_{g-1}+i)\nu_g} \\
&\quad + b_g^{(1)} t^{m_g} + \sum_{i=1}^{\infty} c_i^{(1)} t^{m_g+i}, \\
Y &= \sum_{i=1}^{k_1} a_{1i}^{(2)} t^{\nu_i} + b_1^{(2)} t^{m_1\nu_2} + \cdots + \sum_{i=1}^{k_g} a_{gi}^{(2)} t^{(m_{g-1}+i)\nu_g} \\
&\quad + b_g^{(2)} t^{m_g} + \sum_{i=1}^{\infty} c_i^{(2)} t^{m_g+i},
\end{aligned} \tag{2}
$$

where $a_{11}^{(j)} \neq 0 \neq b_i^{(j)}$ for $j=1, 2$ and $i=1, 2, \cdots, g$.

We prove the following theorem:

THEOREM. *The pairs* (m_j, n_j) *are characteristic pairs of the branch* (2) *if and only if*

$$
\begin{vmatrix} a_{11}^{(1)} & b_j^{(1)} \\ a_{11}^{(2)} & b_j^{(2)} \end{vmatrix} \neq 0, \qquad (j = 1, 2, \cdots, g).
$$

PROOF: Let us make a change in parameter, setting $X=\tau^{\nu}$.

Then we have $t=\alpha\tau+\cdots$, where $\alpha\neq 0$. We first prove that $t\tau^{\nu-1}=[\tau;\ \alpha,\ \beta;\ m_1 n_1\cdots m_\theta n_\theta]$ where $\alpha_{11}=\alpha\neq 0\neq\beta_j$.

Let $t\tau^{\nu-1}=\phi(\tau)$. We first notice that the leading term in τ^ν must occur in $\phi(\tau)$. Let us now suppose $\delta_j\tau^{l_j}$ is the first term of the power series $\phi(\tau)$ whose exponent is not divisible by ν_j. Then we have

$$\begin{aligned} t^\nu = &\ (\text{terms whose exponents are divisible by } \nu_j) \\ &+ (\nu-1)\alpha_{11}^{\nu-1}\delta_j\tau^{l_j} \\ &+ (\text{terms of higher degree in } \tau) \end{aligned}$$

and, in fact, for any positive integer σ,

$$\begin{aligned} t^{\nu_j\sigma} = &\ (\text{terms whose exponents are divisible by } \nu_j) \\ &+ (\nu_j\sigma-1)\alpha_{11}^{\nu_j\sigma-1}\delta_j\tau^{\nu(\nu_j\sigma-\nu)+l_j} \\ &+ (\text{terms of higher degree}). \end{aligned}$$

We also have

$$b_j^{(1)}t^{m_j\nu_{j+1}} = b_j^{(1)}\alpha_{11}^{m_j\nu_{j+1}}\tau^{m_j\nu_{j+1}} + (\text{terms of higher degree}).$$

The contribution of a term $t^{\nu_j\sigma}$ gives for the first term whose exponent is not divisible by ν_j one whose exponent is $\geqq l_j$ and the equality sign holds only for $\nu_j\sigma=\nu$. Since $X=\tau^\nu$, the contribution $(\nu-1)\alpha_{11}^{\nu-1}\delta_j\tau^{l_j}$ must cancel out. It cannot cancel out with any term arising from the contribution of $t^{\nu_j\sigma}$ since it is the term of lowest degree whose exponent is not divisible by ν_j. Therefore, $l_j\geqq m_j\nu_{j+1}$. Also, the contribution $b_j^{(1)}\alpha_{11}^{m_j\nu_{j+1}}\tau^{m_j\nu_{j+1}}$ must cancel out. Now it cannot cancel with any term whose exponent is less than l_j since all such terms have exponents which are divisible by ν_j. Consequently, $m_j\nu_{j+1}\geqq l_j$. On combining these two inequalities, we have the relation $l_j=m_j\nu_{j+1}$, and the statement is established.

On substituting this result in the development for Y, it is seen that the first term which occurs whose exponent is not divisible by ν_j is a term in $\tau^{m_j\nu_{j+1}}$ provided that its coefficient

$$B_j^{(2)} = (\nu-1)a_{11}^{(2)}\alpha_{11}^{\nu-1}\beta_j + b_j^{(2)}\alpha_{11}^{m_j\nu_{j+1}}$$

is distinct from zero.

Thus (m_j, n_j) is a characteristic pair of the branch (2) if and only if $B_j^{(2)}\neq 0$.

Now the development for X must reduce to τ^ν and we must thus have the relation

$$B_j^{(1)} = (\nu - 1)a_{11}^{(1)}\alpha_{11}^{\nu-1}\beta_j + b_j^{(1)}\alpha_{11}^{m_j\nu_j+1} = 0.$$

It therefore follows that (m_j, n_j) is a characteristic pair of (2) if and only if the system of equations

$$\text{(B)} \qquad \left\{\begin{matrix} B_j^{(1)} = 0 \\ B_j^{(2)} = 0 \end{matrix}\right\}$$

possesses no solution. The system (B) cannot be satisfied by the trivial solution since $\alpha_{11} \neq 0 \neq \beta_j$. Consequently, (B) possesses *no* solution if and only if

$$\Delta_j = \begin{vmatrix} a_{11}^{(1)} & b_j^{(1)} \\ a_{11}^{(2)} & b_j^{(2)} \end{vmatrix} \neq 0.$$

This establishes the theorem.

3. *Projections of Space Branches.* Let us now consider a space branch

$$\text{(3)} \qquad \begin{aligned} x &= t^\nu, \\ y &= f(t) = at^\nu + \cdots, \\ z &= \phi(t) = bt^\nu + \cdots. \end{aligned}$$

If we project such a branch from a point (x_0, y_0, z_0) onto a plane, say $z=0$, and consider x_0, y_0, and z_0 as parameters, we obtain a plane branch

$$\text{(4)} \qquad \begin{aligned} x &= x(t; x_0, y_0, z_0), \\ y &= y(t; x_0, y_0, z_0), \end{aligned}$$

in which the coefficients are rational functions of x_0, y_0, z_0 (in fact, polynomials in x_0 and y_0). If we assume that no relations exist between the parameters, (4) will have order ν and certain characteristic pairs (m_j, n_j). These pairs will cease to be characteristic only for special positions of the point (x_0, y_0, z_0) for which the coefficients of the characteristic terms vanish. When no such special relations hold, the center of projection is evidently generic. Obviously, the non-generic centers of projection for a given branch fill up certain algebraic spreads; we shall proceed to the explicit determination of these spreads.

Let us assume that the points at infinity on the three axes are generic with respect to the branch (3). Then the three parallel projections of this branch are generic and we may write its development in the form:

$$
\begin{aligned}
x &= t^{\nu}, \\
y &= \sum_{i=1}^{k_1} a_{1i} t^{\nu_i} + b_1 t^{m_1 \nu_2} + \cdots + \sum_{i=1}^{k_g} a_{gi} t^{(m_{g-1}+i)\nu_g} + b_g t^{m_g} \\
&\quad + \sum_{i=1}^{\infty} c_i t^{m_g+i}, \\
z &= \sum_{i=1}^{k_1} \alpha_{1i} t^{\nu_i} + \beta_1 t^{m_1 \nu_2} + \cdots + \sum_{i=1}^{k_g} \alpha_{gi} t^{(m_{g-1}+i)\nu_g} + \beta_g t^{m_g} \\
&\quad + \sum_{i=1}^{\infty} \gamma_i t^{m_g+i},
\end{aligned} \tag{5}
$$

where, by the preceding section, $\alpha_{11} \neq 0 \neq a_{11}$, $\beta_j \neq 0 \neq b_j$,

$$
\begin{vmatrix} a_{11} & b_j \\ \alpha_{11} & \beta_j \end{vmatrix} \neq 0, \qquad (j = 1, 2, \cdots, g).
$$

Let us project the branch (5) from the point (x_0, y_0, z_0) onto the plane $x=0$. Assume $x_0 \neq 0$, for in the case $x_0 = 0$ the projection is degenerate. The projecting cone is given by

$$
\begin{aligned}
X &= \frac{\rho x_0 + \sigma x}{\rho + \sigma}, \\
Y &= \frac{\rho y_0 + \sigma y}{\rho + \sigma}, \\
Z &= \frac{\rho z_0 + \sigma z}{\rho + \sigma}.
\end{aligned} \tag{6}
$$

Cutting this cone by the plane $X=0$, we obtain the relations

$$
\rho x_0 + \sigma x = 0; \qquad \rho = -\frac{\sigma x}{x_0} = -\frac{\sigma t^{\nu}}{x_0}. \tag{7}
$$

Therefore, the projection of (5) from the point (x_0, y_0, z_0) onto the plane $x=0$ is given by the equations

$$
\begin{aligned}
y &= \frac{[t;\, a,\, b;\, m_1 n_1 \cdots m_g n_g]}{1 - \dfrac{t^r}{x_0}} \\
&= [t;\, a,\, b;\, m_1 n_1 \cdots m_g n_g] \cdot \left(1 + \frac{t^r}{x_0} + \frac{t^{2r}}{x_0^2} + \cdots\right), \\
z &= \frac{[t;\, \alpha,\, \beta;\, m_1 n_1 \cdots m_g n_g]}{1 - \dfrac{t^r}{x_0}} \\
&= [t;\, \alpha,\, \beta;\, m_1 n_1 \cdots m_g n_g] \cdot \left(1 + \frac{t^r}{x_0} + \frac{t^{2r}}{x_0^2} + \cdots\right).
\end{aligned} \tag{8}
$$

We now apply our theorem on plane branches. The characteristic pairs of the projection (8) are accordingly (m_j, n_j) if and only if

$$
D_j = \begin{vmatrix} a_{11} - \dfrac{y_0}{x_0} & b_j \\ \alpha_{11} - \dfrac{z_0}{x_0} & \beta_j \end{vmatrix} \neq 0, \quad (j = 1, 2, \cdots, g).
$$

Thus we have shown that a necessary and sufficient condition that the projection (8) of (5) be generic is that $D_j \neq 0$ for $j = 1, 2, \cdots, g$. It therefore follows that the point (x_0, y_0, z_0) is a generic center of projection for the space branch (5) provided it is not incident to any one of the g planes

$$
P_j\colon \begin{vmatrix} x & y & z \\ 1 & a_{11} & \alpha_{11} \\ 0 & b_j & \beta_j \end{vmatrix} = 0, \quad (j = 1, 2, \cdots, g).
$$

The planes P_j all pass through the tangent line of the branch (5). The osculating plane of (5) is a member of this set if and only if the determinants

$$
\begin{vmatrix} a_{11} & \alpha_{11} \\ a_{1j} & \alpha_{1j} \end{vmatrix}, \qquad (j = 2, 3, \cdots, k_1)
$$

all vanish.

THE JOHNS HOPKINS UNIVERSITY

A NOTE ON THE CESÀRO METHOD OF SUMMATION*

BY J. H. CURTISS

1. *Introduction.* A sequence $\{S_n\}$, or a series $\sum U_n$ with partial sums S_n, is said to be summable by the Cesàro mean of order α, or summable (C, α), to the sum s, if $\sigma_n^\alpha = S_n^\alpha / A_n^\alpha \to s$,† where S_n^α and A_n^α are given by the following relations:

$$(1)\quad (1-x)^{-\alpha-1} = \sum A_n^\alpha x^n; \quad A_n^\alpha = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}{n!};$$

$$(2)\quad \sum S_n^\alpha x^n = (1-x)^{-\alpha}\sum S_n x^n = (1-x)^{-\alpha-1}\sum U_n x^n:$$

$$S_n^\alpha = \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} S_\nu = \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha} U_\nu;$$

and where α is any complex number other than a negative integer.‡ We shall restrict ourselves in this note to real orders of summability. It is known that if a sequence or series S is summable (C, α), $\alpha > -1$, it is summable (C, α'), $\alpha' > \alpha$, to the same sum.§ If a sequence or series S is summable (C, α) for all $\alpha \geqq \gamma$, then the lower limit of all such possible values of γ is called by Chapman‖ the *index of summability* of S.

It is sometimes easier to find the indices of summability and the sums of certain subsequences of a sequence S than to find the index and sum of S itself. As a trivial example, let $\{S_n\}$ be the sequence of partial sums of Leibniz's series $1-1+1-1+\cdots$. Then $S_{2k}=1$, $S_{2k+1}=0$, and it is easily seen that $\{S_{2k}\}$ is summable to the value 1 and $\{S_{2k+1}\}$ to the value 0 by the Cesàro mean of any order. It is the purpose of this note

* Presented to the Society, September 9, 1937.

† Superscripts will not denote exponents when applied to capital letters and to the letter σ.

‡ For a systematic account of the Cesàro method, see Kogbetliantz, *Sommation des Séries et Intégrales Divergentes par les Moyennes Arithmétiques et Typiques*, Paris, 1931.

§ Kogbetliantz, op. cit., p. 17.

‖ Proceedings of the London Mathematical Society, (2), vol. 9 (1911), pp. 369–409; p. 378.

to investigate the relation between the Cesàro sum of a sequence and the Cesàro sums of equally spaced subsequences. We shall also generalize a remark of Chapman concerning diluted series.

2. *The Sum in Terms of the Sums of Subsequences.* Let

$$\sigma_{H,n}^{\alpha} = \frac{\sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} S_{\lambda\nu+H}}{A_n^{\alpha}}, \tag{3}$$

where λ is any positive integer and $0 \leqq H \leqq \lambda - 1$.

THEOREM I. *If for some $\alpha > 0$, $\sigma_{H,n}^{\alpha} \to s_H$, $H = 0, 1, \cdots, \lambda - 1$ (uniformly with respect to any set of parameters (y) on which the terms of the sequence $\{S_n\}$ may depend), then $\sigma_n^{\alpha} \to (\sum_{H=0}^{\lambda-1} s_H)/\lambda$ (uniformly in (y)).*

PROOF. Let K be any integer such that $0 \leqq K \leqq \lambda - 1$, and let $A_k^{\delta} = 0$ for $k < 0$. Then

$$\begin{aligned}
\sigma_{\lambda n+K}^{\alpha} &= \frac{1}{A_{\lambda n+K}^{\alpha}} \sum_{\nu=0}^{\lambda n+K} A_{\lambda n+K-\nu}^{\alpha-1} S_{\nu} \\
&= \frac{1}{A_{\lambda n+K}^{\alpha}} \sum_{H=0}^{\lambda-1} \sum_{\nu=0}^{n} A_{\lambda n+K-(\lambda\nu+H)}^{\alpha-1} S_{\lambda\nu+H} \\
&= \frac{1}{\lambda} \sum_{H=0}^{\lambda-1} \tau_{H,L,n},
\end{aligned}$$

where $L = K - H$ and

$$\tau_n = \tau_{H,L,n} = \frac{\lambda}{A_{\lambda n+K}^{\alpha}} \sum_{\nu=0}^{n} A_{\lambda(n-\nu)+L}^{\alpha-1} S_{\lambda\nu+H}. \tag{4}$$

We shall prove that $\tau_n \to s_H$ uniformly in (y), which will clearly suffice to prove the theorem. Our procedure is essentially this: Let C_{α}^{-1} be the inverse of the matrix of the transformation (3) and let Γ be the matrix of the transformation (4). We show that ΓC_{α}^{-1} satisfies the Silverman-Toeplitz regularity conditions.*

* See, for instance, Carmichael, this Bulletin, vol. 25 (1918), pp. 97–131; p. 109. The method of proof used in this note is discussed in the same reference on pp. 112–113. The author is indebted to R. P. Agnew for the suggestion that the procedure be explained in terms of matrices.

We shall arrange the work in such a way that the order of magnitude of the quantity $\tau_n - s_H$ will be exhibited.

In the sequel, A_n^α will be defined by the first formula of (1) for all values of α, including negative integers. By means of (2), the inverse of the transformation (3) is easily found to be

$$S_{\lambda\nu+H} = \sum_{k=0}^{\nu} A_{\nu-k}^{-\alpha-1} A_k^\alpha \sigma_{H,k}^\alpha.$$

If we replace $\sigma_{H,n}^\alpha$ by $s_H + \epsilon_{H,n}$, substitute in (4), and rearrange terms, we have

$$\tau_n = \frac{s_H \sum_{\nu=0}^{n} \lambda A_{\lambda(n-\nu)+L}^{\alpha-1}}{A_{\lambda n+K}^\alpha} + \frac{\lambda \sum_{\nu=0}^{n} \epsilon_{H,\nu} A_\nu^\alpha B_{\lambda(n-\nu)+L}}{A_{\lambda n+K}^\alpha} = \Sigma_1 + \Sigma_2,$$

where B_n is given by the relation $\sum B_n x^n = (1-x^\lambda)^\alpha (1-x)^{-\alpha}$.

Now let $\Delta_n = \Sigma_1 - s_H$. Then

$$\Delta_n = s_H \frac{\sum_{k=0}^{\lambda-1} \sum_{\nu=0}^{n-1} (A_{\lambda\nu+L}^{\alpha-1} - A_{\lambda\nu+k}^{\alpha-1}) + \lambda A_{\lambda n+L}^{\alpha-1} - \sum_{k=0}^{L} A_{\lambda n+k}^{\alpha-1}}{A_{\lambda n+K}^\alpha}.$$

Letting $\delta_n = \sum_{\nu=0}^{n-1} (A_{\lambda\nu+L}^{\alpha-1} - A_{\lambda\nu+k}^{\alpha-1})$, we prove that $\delta_n = O(1) + O(n^{\alpha-1})$. The result is immediate if $\alpha = 1$. Suppose $\alpha \neq 1$, and $L \geqq k$; the proof is similar if $L < k$. We have

$$\delta_n = \sum_{\nu=0}^{n-1} \sum_{h=0}^{L-k-1} (A_{\lambda\nu+L-h}^{\alpha-1} - A_{\lambda\nu+L-h-1}^{\alpha-1}) = \sum_{\nu=0}^{n-1} \sum_{h=0}^{L-k-1} A_{\lambda\nu+L-h}^{\alpha-2}. \tag{5}$$

We make use of the following lemma of Andersen:*

$$\sum_{\nu=0}^{n} | A_\nu^{-\beta} A_{n-\nu}^{-\gamma} | = O(n^{-\beta}) + O(n^{-\gamma}) + O(n^{-\beta-\nu+1}),$$

provided that $-\beta$, $-\gamma$, and $-\beta-\gamma+1$ are not negative integers. If $-\beta-\gamma+1$ is a negative integer, the equation reads as follows:

$$\sum_{\nu=0}^{n} | A_\nu^{-\beta} A_{n-\nu}^{-\gamma} | = O(n^{-\beta}) + O(n^{-\gamma}).$$

* Andersen, *Studier over Cesàros Summabilitetsmetode*, København, 1922, pp. 22–23.

When the lemma is applied to (5), our assertion concerning the order of δ_n is established at once.

It follows (see (7) below) that

$$\Delta_n = O(n^{-\alpha}) + O(n^{-1}). \tag{6}$$

Equation (6) expresses the fact that the sums of the rows of matrix ΓC_α^{-1} converge to unity, which, of course, is one of the three Silverman-Toeplitz regularity conditions.

Turning to Σ_2, we first observe that $|B_n| \leqq M_1 |A_n^{-\alpha-1}|$, where M_1 is independent of n. The remark is obvious if $\lambda = 2$, or if α is an integer; and in the general case, the reader will have no difficulty in supplying an induction proof based on Andersen's lemma and on the fact that $\sum B_n x^n = \prod_{p=1}^{\lambda=1} (1 - a_p x)^\alpha$, where $|a_p| = 1$. Our second observation is that the well known relation

$$A_n^\delta \cong n^\delta / \Gamma(\delta + 1), \quad \delta \neq -1, -2, \cdots, \tag{7}$$

gives us the inequality $|A_{\lambda n}^\delta| \leqq M_2 |A_n^\delta|$, M_2 independent of n, which is valid for all real values of δ. From these remarks it follows that

$$|\Sigma_2| \leqq M_1 \frac{\sum_{\nu=0}^{n} |\epsilon_{H,\nu} A_\nu^\alpha A_{\lambda(n-\nu)+L}^{-\alpha-1}|}{A_{\lambda n+K}^\alpha}$$

$$\leqq M_1 M_2 \frac{\sum_{\nu=0}^{n} |\epsilon_{H,\nu} A_\nu^\alpha A_{n-\nu}^{-\alpha-1}|}{A_{\lambda n+K}^\alpha}.$$

Now

$$\frac{A_\nu^\alpha A_{n-\nu}^{-\alpha-1}}{A_{\lambda n+K}^\alpha} = o(1) \tag{8}$$

for any fixed value of ν, by (7), and

$$\frac{\sum_{\nu=0}^{n} |A_\nu^\alpha A_{n-\nu}^{-\alpha-1}|}{A_{\lambda n+K}^\alpha} = O(1) \tag{9}$$

by (7) and by the lemma of Andersen if α is fractional, otherwise because $A_n^{-\alpha-1} = 0$ for sufficiently large values of n. It fol-

lows as in the proof of the Silverman-Toeplitz Theorem that if $\epsilon_{H,n} \to 0$ uniformly in (y), Σ_2 behaves likewise. Equations (8) and (9) express the fact that the matrix ΓC_α^{-1} satisfies the remaining two regularity conditions.

3. *Degree of Convergence.* The proof of the theorem is now complete. To summarize, we have shown that whether or not the numbers $\epsilon_{H,n} \to 0$, we have

$$(10)\quad \left| \sigma^{\alpha}_{\lambda n+K} - s \right| \leqq M\left(n^{-\alpha} + n^{-1} + n^{-\alpha} \sum_{H=0}^{\lambda-1} \sum_{\nu=0}^{n} \left| \epsilon_{H,\nu} A_\nu A^{-\alpha-1}_{n-\nu} \right| \right),$$

$(K=0, 1, \cdots, \lambda-1)$, where M is independent of n and K, and $s=(\sum_{H=0}^{\lambda-1} s_H)/\lambda$; and further, that if $\epsilon_{H,n} \to 0$ uniformly in (y), $(H=0, 1, \cdots, \lambda-1)$, then $\sigma_n^\alpha \to s$ uniformly in (y).

By considering the sequence 1, 0, 1, 0, $\cdots$, the reader will have no trouble in showing that the first two terms of our estimate of $\left| \sigma^{\alpha}_{\lambda n+k} - s \right|$ in (10) cannot be improved. If certain restrictions are placed on the sequences $\{\epsilon_{H,n}\}$, it can be proved that the third term in the right hand member of (10) is $O(\epsilon_n)$. We shall amplify this statement only by mentioning two special cases:

(a) If $\sigma^\alpha_{H,n} - s_H = O(n^{-\delta})$, $(H=0, 1, \cdots, \lambda-1)$, uniformly in (y), then $\sigma_n^\alpha - s = O(n^{-1}) + O(n^{-\alpha}) + O(n^{-\delta})$ uniformly in (y).
(b) If $\sigma^\alpha_{H,n} - s_H = O(n^{-\alpha} \log n)$, $(H=0, 1, \cdots, \lambda-1)$, uniformly in (y), then $\sigma_n^\alpha - s = O(n^{-1}) + O(n^{-\alpha} \log n)$.*

4. *The Sum in Terms of the Sums of Subseries.* Chapman called a series "uniformly diluted" if between every pair of terms of the series is placed a constant number of zero terms.† His conjecture‡ that uniform dilution can affect neither the sum nor the summability of a series is a special case of the following theorem.

THEOREM II. *If* $\sum U_{\lambda n+H}$ *is summable* (C, α) *to the sum* u_H, $(H=0, 1, \cdots, \lambda-1)$, *for some* $\alpha > -1$ *(uniformly with respect*

* The author encountered the problems solved in this note while generalizing certain results concerning the Taylor series on the circle of convergence. The importance of the two special cases considered here is due to Szász's work on the degree of summability of the Fourier series, Acta Szeged, vol. 3 (1927), pp. 38–48.

† Loc. cit., p. 404.

‡ Ibid.

to any set of parameters (y) *on which the terms of the series* $\sum U_n$ *may depend), then* $\sum U_n$ *is summable* (C, α) *to the value* $\sum_{H=0}^{\lambda-1} u_H$ *(uniformly in* (y)*).*

The close connection between this theorem and Theorem I is apparent when we notice that

$$\sigma^{\alpha}_{H,n} = \left(\sum_{k=0}^{H} \sum_{\nu=0}^{n} A^{\alpha}_{n-\nu} U_{\lambda\nu+k} + \sum_{k=H+1}^{\lambda-1} \sum_{\nu=0}^{n-1} A^{\alpha}_{n-\nu} U_{\lambda\nu+k} \right) / A^{\alpha}_{n}.$$

Since the conclusion of Theorem I is not true for $\alpha \leqq 0$ (see §5 below), a general proof is more conveniently given by the methods of §2 than by reference to this theorem. Equation (3) is to be replaced by the transformation $\sigma^{\alpha}_{H,n} = (\sum_{\nu=0}^{n} A^{\alpha}_{n-\nu} U_{\lambda\nu+H}) / A^{\alpha}_{n}$ and (4) by the transformation $\tau_n' = (\sum_{\nu=0}^{n} A^{\alpha}_{\lambda(n-\nu)+L} U_{\lambda\nu+H}) / A^{\alpha}_{\lambda n+k}$. The reader will have no difficulty in filling in the details.

5. *Discussion.* Of course the converse of neither Theorem I nor Theorem II is true. In certain senses the theorems cannot be strengthened. In the first place, the index of summability of a sequence or series may be equal to the greatest of the indices of the subsequences or subseries. An example is the series $1+0-2+0+3+0-4+0+\cdots$, whose index is 1. (We remark furthermore that this series is not summable $(C, 1)$.) Again, the conclusion of Theorem I is false for $\alpha<0$, even if the sums s_H be equal, as the reader may show by examining the sequence $0, A_0^{\delta}, 0, A_1^{\delta}, 0, A_2^{\delta}, \cdots$, where $-1<\alpha<\delta<0$. Finally, the conclusion of Theorem II is false for $\alpha<-1$; an example is the series $0+A_0^{\delta}+0+A_1^{\delta}+0+A_2^{\delta}+\cdots$, where $-2<\alpha<\delta<-1$.

CORNELL UNIVERSITY

INEQUALITIES SATISFIED BY A CERTAIN DEFINITE INTEGRAL

BY G. H. HARDY AND NORMAN LEVINSON*

1. *Introduction.* In this note we solve the following problem. Suppose that

$$0 \leqq a_1 < a_2 < \cdots < a_{2n+1} \leqq 1,$$

$$(1) \qquad f(x) = \frac{(x-a_2)(x-a_4)\cdots(x-a_{2n})}{(x-a_1)(x-a_3)\cdots(x-a_{2n+1})},$$

$$J(t) = \int_0^1 |f(x)|^t dx, \qquad 0 < t < 1.$$

Then what are the best inequalities satisfied by $J(t)$?

We prove the following theorem:

THEOREM A. *If $f(x)$ satisfies* (1) *then*

$$\frac{\Gamma(\frac{1}{2}+\frac{1}{2}t)\Gamma(1-\frac{1}{2}t)}{(1-t)\pi^{1/2}} \leqq J(t) \leqq \frac{2^t}{1-t},$$

with inequality except when

$$f(x) = \frac{1}{x-\frac{1}{2}}, \qquad J(t) = \frac{2^t}{1-t};$$

$$f(x) = \frac{x-\frac{1}{2}}{x(x-1)}, \qquad J(t) = \frac{\Gamma(\frac{1}{2}+\frac{1}{2}t)\Gamma(1-\frac{1}{2}t)}{(1-t)\pi^{1/2}}.$$

The integral $J(t)$ occurred in a recent paper by Levinson.† Levinson proved that

$$J(t) < \frac{5}{1-t},$$

and indeed that

$$\int_0^1 |f(x+iy)|^t < \frac{5}{1-t}$$

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† Levinson, *On non-harmonic Fourier series*, Annals of Mathematics, (2), vol. 37 (1936), p. 922.

for any real y, and stated without proof a more precise, though still not the best possible, inequality. Here we confine ourselves to the case $y=0$, but our results are the best of their kind. We prove them by two methods, one "real" and one "complex".

2. *A Theorem of Boole.* LEMMA 1. *If $f(x)$ satisfies* (1), *then*

$$\int_{-\infty}^{\infty} F\{f(x)\}\,dx = \int_{-\infty}^{\infty} F(y)\,\frac{dy}{y^2} \tag{2}$$

whenever (i) *$F(y)$ is defined for all values of y, and* (ii) *either integral exists as a Lebesgue integral.*

Lemma 1 is essentially the same as a theorem of Boole.*

There are two other definitions of $f(x)$ equivalent to that of §1. In the first place, as we can verify at once by resolving $f(x)$ into partial fractions,

$$f(x) = \sum_{\nu=0}^{n} \frac{\alpha_\nu}{x - a_{2\nu+1}}, \tag{3}$$

where

$$\alpha_\nu > 0, \qquad \sum \alpha_\nu = 1. \tag{4}$$

This is the form which we shall generally use here. Secondly

$$g(x) = \frac{1}{f(x)} = x - \sum_{\nu=0}^{n} \frac{\beta_\nu}{x - a_{2\nu}},$$

where $\beta_\nu > 0$. If we write $1/y$ for y and $G(y)$ for $F(1/y)$, then (2) becomes

$$\int_{-\infty}^{\infty} G\{g(x)\}\,dx = \int_{-\infty}^{\infty} G(y)\,dy,$$

which is Boole's formula.

To prove Lemma 1 we observe that, after (3) and (4), the graph of $f(x)$ consists of $n+2$ descending pieces corresponding to the intervals $(-\infty, a_1)$, (a_1, a_3), $\cdots$, (a_{2n+1}, ∞), the corresponding intervals of variation of $f(x)$ being $(0, -\infty)$,

* G. Boole, *On the comparison of transcendents, with certain applications to the theory of definite integrals*, Philosophical Transactions of the Royal Society, vol. 147 (1857), pp. 745–803. See in particular p. 780. Boole's very interesting memoir has been forgotten, and his results have been rediscovered, wholly or in part, by a number of later mathematicians.

$(\infty, -\infty), \cdots, (\infty, 0)$; and that, when x moves from $-\infty$ to ∞, y moves, in all, $n+1$ times over the same range. The line $f(x)=y$ cuts the graph of $f(x)$ in $n+1$ points $x_1, x_2, \cdots, x_{n+1}$; and

$$\int_{-\infty}^{\infty} F(y)dx = \int_{-\infty}^{\infty} F(y)P(y)\frac{dy}{y^2},$$

where

$$P(y) = -y^2 \sum_{\nu} \left(\frac{dx}{dy}\right)_{x=x_\nu}.$$

We have to prove that*

$$P(y) = 1.$$

It is plain that, if $f(x)=y$, then

$$(5) \qquad y\prod_{\nu}(x - a_{2\nu+1}) - \sum_{\nu} \alpha_\nu \prod_{\mu\neq\nu}(x - a_{2\mu+1}) = y\prod_{\nu}(x - x_\nu).$$

Hence, first, equating the coefficients of x^{n-1} and using (4), we have

$$(6) \qquad \sum x_\nu - \sum a_{2\nu+1} = \frac{1}{y}.$$

Next, (6) is an identity in y when $x_\nu(y)$ is substituted for x_ν. Hence, differentiating this, we obtain

$$\sum \frac{dx_\nu}{dy} = -\frac{1}{y^2}.$$

It follows that $P(y)=1$.

3. *The Underlying Identity.* In what follows it is convenient to symmetrize our analysis about the origin, which we can do by writing $x-\frac{1}{2}$ for x. We have then

$$(7) \quad J(t) = \int_{-1/2}^{1/2} |f(x)|^t dx, \quad f(x) = \sum \frac{\alpha_\nu}{x - a_{2\nu-1}}, \quad \alpha_\nu > 0, \quad \sum \alpha_\nu = 1,$$

and

$$(8) \qquad -\tfrac{1}{2} \leqq a_1 < a_2 < \cdots < a_{2n+1} \leqq \tfrac{1}{2}.$$

* We are indebted to Professor Bohnenblust for a simplification of the proof.

LEMMA 2. *If $f(x)$ satisfies* (7) *and* (8), *then*

$$J(t) = \frac{2^t}{1-t} - \int_{1/2}^{\infty} \left\{ |f(x)|^t + |f(-x)|^t - \frac{2}{x^t} \right\} dx. \tag{9}$$

Suppose that ϵ is small and positive and that ξ and η are the largest and smallest roots of $f(x) = \epsilon$ and $f(x) = -\epsilon$ respectively. Then $\xi > \frac{1}{2}$ and $\eta < -\frac{1}{2}$. Also

$$\frac{1}{\xi + \frac{1}{2}} \leqq \sum \frac{\alpha_\nu}{\xi - a_{2\nu+1}} = \epsilon \leqq \frac{1}{\xi - \frac{1}{2}},$$

and so

$$\frac{1}{\epsilon} - \frac{1}{2} \leqq \xi \leqq \frac{1}{\epsilon} + \frac{1}{2},$$

$$\xi = \frac{1}{\epsilon} + O(1), \tag{10}$$

where the O refers to the limit process $\epsilon \to 0$. Similarly

$$\eta = -\frac{1}{\epsilon} + O(1). \tag{11}$$

Define f_ϵ by the relations

$$f_\epsilon = f, \quad (|f| \geqq \epsilon); \qquad f_\epsilon = 0, \quad (|f| < \epsilon).$$

Then, by Lemma 1,

$$\int_{-\infty}^{\infty} |f_\epsilon|^t dx = 2 \int_{\epsilon}^{\infty} |y|^{t-2} dy = \frac{2\epsilon^{t-1}}{1-t}.$$

Hence

$$\begin{aligned} J(t) &= \int_{-1/2}^{1/2} |f|^t dx = \lim_{\epsilon \to 0} \int_{-1/2}^{1/2} |f_\epsilon|^t dx \\ &= \lim_{\epsilon \to 0} \left\{ \frac{2\epsilon^{t-1}}{1-t} - \left(\int_{1/2}^{\xi} |f|^t dx + \int_{\eta}^{-1/2} |f|^t dx \right) \right\}. \end{aligned} \tag{12}$$

Now

$$f(x) = x^{-1} + O(x^{-2}), \qquad |f(x)|^t = |x|^{-t} + O(|x|^{-t-1})$$

for large x. Hence, by (10),

$$\int_{1/\epsilon}^{\xi} |f|^t dx = \frac{1}{1-t}\left\{\left(\frac{1}{\epsilon}+O(1)\right)^{1-t} - \left(\frac{1}{\epsilon}\right)^{1-t}\right\} + O(\epsilon^t)$$
$$= O(\epsilon^t),$$

and we may replace ξ by $1/\epsilon$ in (12). Similarly we may replace η by $-1/\epsilon$. Hence

$$J(t) = \lim_{\epsilon\to 0}\left\{\frac{2\epsilon^{t-1}}{1-t} - \int_{1/2}^{1/\epsilon}\{|f(x)|^t + |f(-x)|^t\}dx\right\}$$
$$= \lim_{\epsilon\to 0}\left\{\frac{2\epsilon^{t-1}}{1-t} - 2\int_{1/2}^{1/\epsilon}\frac{dx}{x^t} - \int_{1/2}^{1/\epsilon}\left\{|f(x)|^t + |f(-x)|^t - \frac{2}{x^t}\right\}dx\right\},$$

which is (9).

4. *A Lemma.* LEMMA 3. *If* $|x| > \frac{1}{2}$ *then*

$$\phi(x) = |f(x)|^t + |f(-x)|^t$$

is (for every x*) least and greatest when* $f(x)$ *is* $1/x$ *and* $x/(x^2-\frac{1}{4})$ *respectively.*

We may suppose $x > \frac{1}{2}$. We consider the pole A of $f(x)$ nearest to an end of $(-\frac{1}{2}, \frac{1}{2})$. If we suppose, for example, that $A > 0$, then $A = a_{2n+1}$. If

$$\xi = \frac{1}{x-a}, \quad \xi' = \frac{1}{x+a}, \quad \Xi = \frac{1}{x-A}, \quad \Xi' = \frac{1}{x+A},$$

then all these numbers are positive and

$$\frac{\Xi}{\Xi'} > \frac{\xi}{\xi'} \geqq 1 \tag{13}$$

for any pole a other than A. If

$$\psi(A) = \phi(x) = |f(x)|^t + |f(-x)|^t$$
$$= \left(\sum\frac{\alpha}{x-a}\right)^t + \left(\sum\frac{\alpha}{x+a}\right)^t,$$

then

$$\frac{1}{t}\frac{d\psi(A)}{dA} = |f(x)|^{t-1}\frac{A}{(x-A)^2} - |f(-x)|^{t-1}\frac{A}{(x+A)^2},$$

where A is the α corresponding to A. This will be positive if

$$\left(\frac{\Xi}{\Xi'}\right)^2 > \left(\frac{\sum \alpha\xi}{\sum \alpha\xi'}\right)^{1-t},$$

and this is true on account of (13).

Hence we decrease $\phi(x)$ by moving A to the left, to the next pole, or to the origin if there is no other positive pole. Similarly, if A were negative, we should decrease $\phi(x)$ by moving A to the right. It follows by repetition of the argument that $\phi(x)$ is least when all the a's coincide at the origin, and $f(x)=1/x$.

Similarly $\phi(x)$ is greatest when all the a's are at one of the ends of $(-\frac{1}{2}, \frac{1}{2})$. In this case

$$f(x) = \frac{\alpha}{x-\frac{1}{2}} + \frac{1-\alpha}{x+\frac{1}{2}} = \frac{x-\beta}{x^2-\frac{1}{4}},$$

where $\beta=\alpha-\frac{1}{2}$, $0\leqq\alpha\leqq 1$, $|\beta|\leqq\frac{1}{2}$. Finally

$$|x-\beta|^t + |x+\beta|^t < 2|x|^t$$

if $|x|>\frac{1}{2}$, $\beta\neq 0$, so that the true maximum of $\phi(x)$ occurs when

$$f(x) = \frac{x}{x^2-\frac{1}{4}}.$$

5. *Proof of the Inequalities.* We can now prove the theorem. We take the interval as $(-\frac{1}{2}, \frac{1}{2})$, so that the two critical functions are

$$f_1(x) = \frac{1}{x}, \qquad f_2(x) = \frac{x}{x^2-\frac{1}{4}}.$$

It follows from (9) and Lemma 3 that

$$J(t) \leqq \frac{2^t}{1-t},$$

with inequality unless $f=f_1$. Also

$$\int_{-1/2}^{1/2} |f|^t dx - \int_{-1/2}^{1/2} |f_2|^t dx$$

$$= \int_{1/2}^{\infty} (|f_2(x)|^t + |f_2(-x)|^t - |f(x)|^t - |f(-x)|^t) dx,$$

by (9), and the last integral is positive, by Lemma 3, unless $f=f_2$. Finally

$$\int_{-1/2}^{1/2} |f_2|^t dx = \int_{-1/2}^{1/2} \left| \frac{x}{x^2 - \frac{1}{4}} \right|^t dx = \frac{\Gamma(\frac{1}{2} + \frac{1}{2}t)\Gamma(1 - \frac{1}{2}t)}{(1-t)\pi^{1/2}},$$

by an elementary calculation.

6. *Alternative Proof of the Underlying Identity.* There is another proof of (9) by complex integration. We integrate

$$\int \left\{ (f(x))^t - \frac{1}{x^t} \right\} dx$$

around a contour C composed of (i) small semicircles of radius ρ, above the real axis, around the singularities a_k and 0, (ii) a large semicircle of radius R, above the real axis, around 0, and (iii) the parts of the real axis between these semicircles. We suppose

$$(f(x))^t > 0, \qquad x^t > 0$$

for large positive x, and make $\rho \to 0$ and $R \to \infty$ in the usual manner. Then $(f(x))^t$ is positive along

$$(a_1, a_2)\ (a_3, a_4), \cdots, (a_{2n+1}, \infty)$$

and has the argument of $e^{-t\pi i}$ on the rest of the axis, while x^t is positive for $x>0$ and has the argument of $e^{-t\pi i}$ for $x<0$. We thus obtain

$$I_1(t) + e^{-t\pi i} I_2(t) = 0, \tag{14}$$

where

$$I_1(t) = \left(\int_{a_1}^{a_2} + \int_{a_3}^{a_4} + \cdots + \int_{a_{2n-1}}^{a_{2n}} + \int_{a_{2n+1}}^{1/2} \right) |f(x)|^t dx$$

$$- \int_0^{1/2} \frac{dx}{|x|^t} + \int_{1/2}^{\infty} \left(|f(x)|^t - \frac{1}{|x|^t} \right) dx,$$

$$I_2(t) = \left(\int_{-1/2}^{a_1} + \int_{a_2}^{a_3} + \cdots + \int_{a_{2n}}^{a_{2n+1}}\right) |f(x)|^t dx$$
$$- \int_{-1/2}^{0} \frac{dx}{|x|^t} + \int_{-\infty}^{-1/2} \left(|f(x)|^t - \frac{1}{|x|^t}\right) dx.$$

If we equate imaginary parts in (14) we obtain

$$\left(\int_{-1/2}^{a_1} + \int_{a_2}^{a_3} + \cdots + \int_{a_{2n}}^{a_{2n+1}}\right) |f(x)|^t dx$$
$$= \frac{2^{t-1}}{1-t} - \int_{1/2}^{\infty} \left(|f(-x)|^t - \frac{1}{|x|^t}\right) dx;$$

and if we multiply by $e^{t\pi i}$, and equate imaginary parts, we obtain

$$\left(\int_{a_1}^{a_2} + \int_{a_3}^{a_4} + \cdots + \int_{a_{2n+1}}^{1/2}\right) |f(x)|^t dx$$
$$= \frac{2^{t-1}}{1-t} - \int_{1/2}^{\infty} \left(|f(x)|^t - \frac{1}{|x|^t}\right) dx.$$

Finally (9) follows by addition.

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NOTE ON A THEOREM CHARACTERIZING GEODESIC ARCS IN COMPLETE, CONVEX METRIC SPACES

BY L. M. BLUMENTHAL

1. *Introduction.* In his four *Untersuchungen über allgemeine Metrik*,* Menger initiated the systematic study of the metric geometry of abstract semi-metric and metric spaces. Among the most important of the notions Menger introduced in such spaces is that of *convexity* which leads, in complete metric spaces, to the existence of geodesic arcs joining each pair of points a, b of the space. Such an arc is congruent to a line segment of length ab.†

Concerning geodesic arcs in complete, convex metric spaces, Menger gives the following theorem:‡

THEOREM. *The geodesic arcs joining two points a, b of a complete, convex metric space are characterized among all arcs joining a, b by the following property: if p, q are elements of a geodesic arc joining a, b (p, q both distinct from a, b) then either p is between§ a and q, or p is between q and b, or p is identical with q.*

That a geodesic arc joining a, b has this property follows directly, as Menger observes, from the fact that such an arc may be imbedded congruently in a line segment of length ab. To show, however, that the property is *characteristic* for geodesic arcs it must be shown, of course, that every arc joining a, b that has this property is a geodesic arc. This sufficiency of the property is not shown by Menger (two proofs of the *necessity* of the condition being given instead). As the theorem is of use in developing some of the properties of convex spaces, and as a search of the literature, as well as conversation with Menger, has re-

* Mathematische Annalen, vol. 100 (1928), pp. 75–163; vol. 103 (1930), pp. 466–501.

† The term "geodesic arc" is used by the author in the geometry of distances always in the sense of minimizing geodesic.

‡ *Erste Untersuchung*, loc. cit., p. 91.

§ A point q lies between two points p, r if and only if $p \neq q \neq r$ and $pq + qr = pr$. We symbolize this relation by writing pqr.

vealed that apparently no previous notice of this lacuna has occurred, it seems worthwhile to fill the indicated gap by supplying a short proof of the sufficiency of the property.

2. *The Theorem.* We wish to prove the following theorem:

THEOREM. *If B is an arc joining two points a, b of a complete, convex metric space, and if B has the property that for every pair of its points p, q (distinct from a, b) either apq or qpb or $p=q$ exists, then B is a geodesic arc.*

PROOF. We assume that B is not a geodesic arc and deduce a contradiction.

From the assumption that B is not a geodesic arc it follows that B contains at least one pair of points without containing a middle point of the pair.* For if this were not the case, then B, being a closed, compact subset of a metric space, and containing a middle point for each pair of its elements, would, by the Existence Theorem for geodesic arcs,† contain a geodesic arc B' joining a, b. But since B is itself an arc with endpoints a, b, it is clear that B would be identical with B', which contradicts our assumption that B is not a geodesic arc.

Further, it is easy to show that B contains at least one pair of points b_0, b_1, *each distinct from* a, b, without containing a middle point of the pair; for a denial of this assertion leads at once (because of the continuity of the metric and the compactness of B) to the conclusion that B contains a middle point for each pair of its elements, which, as we have seen, violates the assumption that B is not a geodesic arc.

Though B does not contain a middle point of b_0, b_1, that subarc of B joining b_0, b_1 does contain a point c which is equidistant from b_0 and b_1; that is, c satisfies the relations

$$(1) \qquad b_0c = cb_1 > 0; \qquad b_0c + cb_1 > b_0b_1.$$

It follows that the three points b_0, c, b_1 are not linear.‡

Apply, now, the property stated in the theorem to each of

* A point q is a middle point of two points p, r if and only if q is between p and r, and q is equidistant from p, r.

† *Erste Untersuchung*, loc. cit., p. 89.

‡ Three pairwise distinct points are linear if and only if one of the points is between the other two.

the pairs of *distinct* points b_0, b_1; c, b_0; b_1, c in turn. We obtain the result that ab_0b_1 or b_1b_0b *and* acb_0 or b_0cb *and* ab_1c or cb_1b exist. But it is readily verified that no one of the eight possible combinations can exist, for assuming any one of them leads, by an application of the transitive property of the betweenness relation in metric spaces, to the conclusion that the triple b_0, c, b_1 is linear, in violation of the relations (1) above.* This completes the proof of the theorem.

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A TRANSFORMATION ASSOCIATED WITH THE TRISECANTS OF A RATIONAL TWISTED QUINTIC CURVE†

BY L. A. DYE

1. *Introduction.* This transformation is generated by the use of a (1, 1) correspondence between a pencil of ruled cubic surfaces $|F|$, and their simple directrices which are the trisecants of a rational twisted quintic curve C_5. All of the cubic surfaces contain C_5 and have the quadrisecant of C_5 as a double line l. Through a general point P of space passes one F whose simple directrix r determines with P a plane tangent to the ruled surface of trisecants of C_5 at a point Q on r. The line PQ meets F in a residual point P' which is the image of P in an involutorial Cremona transformation of order 43. A special feature of the transformation is the existence of two ruled surfaces whose generators are parasitic lines of the transformation. One of these surfaces is a principal surface, and the other is not.

Other transformations generated by a somewhat similar method have been discussed by the author in recent papers.‡

2. *The Pencil of Cubic Surfaces.* The equation of a pencil of cubic surfaces, parameter λ, with a double line $l \equiv x_1 = x_2 = 0$, is

* *Erste Untersuchung*, loc. cit., p. 78. For example, the combination ab_0b_1, acb_0, ab_1c cannot exist since the second and third of these relations imply the existence of b_1cb_0.

† Presented to the Society, December 29, 1936.

‡ This Bulletin, vol. 42 (1936), pp. 535–540. Tôhoku Mathematical Journal, vol. 43, (1937), pp. 174–177.

$$F(x) \equiv \lambda F'(x) + F''(x) \equiv x_1x_2(a_1x_1 + a_2x_2) + Bx_3 + Cx_4 = 0,$$

where

$$B \equiv b_{11}x_1^2 + b_{12}x_1x_2 + b_{22}x_2^2,\ C \equiv c_{11}x_1^2 + c_{12}x_1x_2 + c_{22}x_2^2,$$
$$a_i \equiv \lambda a_i' + a_i'', \qquad b_{ij} = \lambda b_{ij}' + b_{ij}'', \qquad c_{ij} = \lambda c_{ij}' + c_{ij}''.$$

The equations of the simple directrix r of $F(x)$ are

$$u \equiv b_{11}D_1x_1 - b_{22}D_2x_2 - Dx_4 = 0,$$
$$v \equiv c_{11}D_1x_1 - c_{22}D_2x_2 + Dx_3 = 0,$$

where

$$D_i \equiv a_1A_{1i} + a_2A_{2i}, \qquad D \equiv A_{12}^2 - A_{11}A_{22},$$
$$A_{11} \equiv b_{12}c_{22} - b_{22}c_{12}, A_{12} \equiv b_{22}c_{11} - b_{11}c_{22}, A_{22} \equiv b_{11}c_{12} - b_{12}c_{11}.$$

If $F(x)$ is multiplied by D it may be factored as follows: $DF(x) \equiv Bv - Cu$.

Since the line r is a trisecant of C_5 (the residual intersection of two surfaces $F(x)$) the eliminant of $u = v = 0$ is the equation of the ruled surface of trisecants, R. This surface is of order 8 since u, v are each of order 4 in λ. The coordinates of a tangent plane to R at any point (ξ) on r are the values at (ξ) of the coefficients of dx_i in the form obtained by eliminating $d\lambda$ from the total differentials du, dv. If the plane passes through a general point $P(y)$ of space, the coordinates of the point of contact $Q(z)$ with R are*

$$z_1 = D(\gamma_2u - \beta_2v),$$
$$z_2 = D(\gamma_1u - \beta_1v),$$
$$z_3 = c_{22}D_2(\gamma_1u - \beta_1v) - c_{11}D_1(\gamma_2u - \beta_2v),$$
$$z_4 = -b_{22}D_2(\gamma_1u - \beta_1v) + b_{11}D_1(\gamma_2u - \beta_2v),$$

where

$$\beta_i \equiv D_i(b_{ii}'D'' - b_{ii}''D') - D(b_{ii}'D_i'' - b_{ii}'D_i'),$$
$$\gamma_i \equiv D_i(c_{ii}'D'' - c_{ii}''D') - D(c_{ii}'D_i'' - c_{ii}''D_i'),$$
$$D' \equiv \partial D/\partial\lambda, D'' \equiv 4D - \lambda D', D_i' \equiv \partial D_i/\partial\lambda, D_i'' \equiv 3D_i - \lambda D_i'.$$

* From here on the variable y_i will replace x_i in all forms and no change in notation will be made.

The residual intersection of the line PQ with the surface F is the point $P'(y')$ whose coordinates are

$$y_i' = y_i(y_i\partial F(z)/\partial z_i) - z_i(z_i\partial F(y)/\partial y_i).$$

3. *An Involution on l.* Any plane through r cuts two residual lines from a surface F which with l determine a conjugate pair of planes in an involution I on l. The equations of this involution are

$$y_1' = A_{12}y_1 - A_{11}y_2, \qquad y_2' = A_{22}y_1 - A_{12}y_2.$$

The planes determined by l and the two torsal generators of F are the double planes of I. They are

$$K \equiv A_{22}y_1^2 - 2A_{12}y_1y_2 + A_{11}y_2^2 = 0.$$

The three planes through l and the tangents to C_5 at its three points on r are

$$DT \equiv y_2(B\gamma_2 - C\beta_2) - y_1(B\gamma_1 - C\beta_1) = 0.$$

The associated triad of planes in I is

$$L \equiv (A_{22}y_1 - A_{12}y_2)(B\gamma_2 - C\beta_2) - (A_{12}y_1 - A_{11}y_2)(B\gamma_1 - C\beta_1).$$

The involution fails when $D=0$. The four cubic surfaces which correspond to these values of λ are the Cayley surfaces of the pencil, that is, they are surfaces in which the simple directrix r is adjacent to l, and along l there is a fixed tangent plane to each surface.

4. *The Involutorial Space Transformation.* If λ is replaced by its value $-F''(y)/F'(y)$, in the equations defining (y'), these become the equations of a space transformation.* The forms u, v are now factorable into BR, CR respectively, and the polars of F factor as follows: $y_i\partial F(z)/\partial z_i \equiv TD^3R^3L$, and $z_i\partial F(y)/\partial y_i \equiv TD^2R^2K$. This results in the factors TD^2R^2 coming off from the transformation, and its equations now take the form

* In order to avoid a confusing number of new symbols the same letters used heretofore to represent functions of (λ, y) will now be used to represent the same functions after the substitution for λ.

$$y_1' = (A_{12}y_1 - A_{11}y_2)TD^2,$$
$$y_2' = (A_{22}y_1 - A_{12}y_2)TD^2,$$
$$y_3' = vL - TD[c_{11}D_1(A_{12}y_1 - A_{11}y_2) - c_{22}D_2(A_{22}y_1 - A_{12}y_1)],$$
$$y_4' = -uL + TD[b_{11}D_1(A_{12}y_1 - A_{11}y_2) - b_{22}D_2(A_{22}y_1 - A_{12}y_1)].$$

It is seen from the equations of the transformation that the four Cayley surfaces taken twice, and the surface T of order twelve, are the images of the line l. Geometrically this can be explained in the following manner. Any plane through l cuts a Cayley surface in a generator s. The point of contact Q of the plane with R is on l, and each point of $s \sim Q$ or $Q \sim s$. The surface T is generated by the tangents to C_5 at points on a trisecant r. Such a tangent line passes through the point of contact Q of the tangent plane to R, so that each point of the line corresponds to the whole line. The line is parasitic and the surface T is a factor of the transformation, but the tangent line meets l and the image of the point on l is the tangent line and, as the image of l, the surface T stays in the transformation as a principal surface.

The plane tangent to R at a point on C_5 on a trisecant r cuts a residual line from F in addition to the trisecant and the tangent. These residual lines are the images of the points on C_5. They generate the surface L of order 30. In any plane through r the points of C_5 not on r go into points on the two secants cut from F. These image points generate a surface G of order 24 which is an additional image of C_5. The form of this surface is most easily obtained by subjecting an F to the transformation $F \sim 4D+2T+L+G+F$.

When a plane through r passes through a torsal generator of F, then each point of the generator corresponds to itself, since the lines joining such points to the point of contact Q are tangent to F. These tangent lines generate the invariant octavic surface, K.

Each generator of T determines two generators of K, and each generator of K determines three generators of T. In this (2, 3) correspondence there are five coincidences, that is, there are five torsal generators of surfaces F which are tangent to C_5 at a

point on the trisecant of the surface. These five lines l_i are parasitic double lines on the homoloidal surfaces.

The only remaining fundamental lines are the trisecants of C_5. These lines are all parasitic and the surface R is shed off the transformation and does not appear as a principal surface.

5. *Table of Characteristics.* The following table exhibits the nature of the images of the several fundamental elements of the transformation.

$$
\begin{aligned}
S_1 &\sim S_{43}{:}l^{30}4\bar{l}^{3}C_5^{12+t}5l_i^{2},\\
l &\sim 2D_{12}{:}l^{8}4\bar{l}C_5^{4} + T_{12}{:}l^{9}C_5^{2+t}5l_i,\\
C_5 &\sim l^{22}4\bar{l}C_5^{9}5l_i + G_{24}{:}l^{16}4\bar{l}^{4}C_5^{7+t}5l_i^{3},\\
K_8 &\sim K_8{:}l^{6}4\bar{l}C_5^{2}5l_i,\\
&\quad R_8{:}l^{4}C_5^{3+t}5l_i.
\end{aligned}
$$

The notation $4\bar{l}$ is used to indicate that four sheets of the surface have contact along l with the four Cayley surfaces. The notation C_5^{t} denotes contact along C_5 due to the fact that R and T have the same tangent planes at each point of C_5. In the total intersection of K and S_{43} the line l must be counted an additional four times since it plays the role of an invariant line in the four fixed tangent planes to the Cayley surfaces. The complete intersection of two homoloidal surfaces may be set down as follows

$$[S_{43}, S'_{43}] = l^{900} + 4\bar{l}^{9} + C_5^{169+1} + 5l_i^{4} + C_{43}.$$

THE CITADEL

THE RESULTANT MATRIX OF TWO POLYNOMIALS*

BY M. M. FLOOD

1. *Introduction.* Frobenius† has shown that if P is a matrix whose characteristic function is $P(x)$ and if $P_0(x)$ is a second polynomial, then their resultant is the determinant of the matrix $P_0(P)$. In particular, if P is non-derogatory,‡ the present author§ has shown that the degree of the highest common factor of $P(x)$ and $P_0(x)$ is the same as the nullity‖ of $P_0(P)$.

In this paper the matrix P is taken to be the companion matrix¶ of $P(x)$, and it is shown that all the remainders in the euclidean algorithm for $P(x)$ and $P_0(x)$ can easily be found from the "resultant matrix" $P_0(P)$. The proof is strictly rational and quite elementary. Finally, the results are applied to a numerical example.

2. *The Algorithm.* The euclidean algorithm for the polynomials $P_0(x)$ and $P_1(x) = P(x)$ may be written in the form

$$(1) \qquad P_{k-1}(x) = R_k(x)P_k(x) - P_{k+1}(x), \qquad (k = 1, 2, \cdots, r),$$

where $P_{r+1}(x) = 0$, and the degree of $P_{k+1}(x)$ is less than the degree of $P_k(x)$. Set $S_1(x) = 1$, $S_2(x) = R_1(x)$, $P_{-1}(x) = 0$, $P_{-2}(x) = 1$, and define polynomials $S_k(x)$ and $P_{-k}(x)$ by the relations

$$\left.\begin{aligned} S_{k+1}(x) &= R_k(x)S_k(x) - S_{k-1}(x) \\ P_{-(k+1)}(x) &= R_k(x)P_{-k}(x) - P_{-(k-1)}(x) \end{aligned}\right\}, \quad (k = 2, 3, \cdots, r).$$

A simple induction** now yields the identities

$$(2) \quad P_k(x) = S_k(x)P_1(x) - P_{-k}(x)P_0(x), \quad (k = 1, 2, \cdots, r + 1).$$

* Presented to the Society, February 29, 1936. A special case of the principal result of this paper was considered by the present author in a paper having the same title and published in the American Mathematical Monthly, vol. 44 (1937), p. 309.

† Frobenius, Journal für Mathematik, vol. 84 (1878), p. 11.

‡ Sylvester has called a matrix "non-derogatory" when its characteristic function and minimum function are the same.

§ American Mathematical Monthly, vol. 43 (1936), p. 562.

‖ The "nullity" of a matrix is the difference between its order and rank

¶ The "companion matrix" of $P(x)$ is the matrix P_1 defined in §3.

** Netto, *Vorlesungen über Algebra*, §62.

For convenience, set

$$P_j(x) = \sum_{k=0}^{p_j} P_{jk}x^k, \qquad (|\,j\,| = 0, 1, 2, \cdots, r+1)$$

and suppose that $p_0 \geqq p_1$ and that $P_{1p_1} = 1$. If $R_k(x)$ is of degree $r_k = r_{-k}$, then

$$r_k = p_{k-1} - p_k = p_{-(k+1)} - p_{-k}, \qquad (|\,k\,| = 2, 3, \cdots, r),$$

and it follows that

$$p_{-k} = p_1 - p_{k-1}, \qquad (|\,k\,| = 2, 3, \cdots, r+1).$$

The euclidean algorithm for the polynomials $P_0(x) = Q_0(x)$ and $P_1(x) = Q_1(x)$ written in the customary form would be

$$Q_{k-1}(x) = \overline{R}_k(x)Q_k(x) + Q_{k+1}(x), \qquad (k = 1, 2, \cdots, r),$$

where $Q_{r+1}(x) = 0$, and the degree of $Q_{k+1}(x)$ is less than the degree of $Q_k(x)$. It is possible to pass easily from one form of the algorithm to the other with the help of the relations

$$\left.\begin{aligned} Q_k(x) &= (-1)^{k(k-1)/2}P_k(x) \\ \overline{R}_k(x) &= (-1)^{k-1}R_k(x) \end{aligned}\right\}, \qquad (k = 0, 1, 2, \cdots, r+1).$$

3. *The Companion Matrix.* The companion matrix of $P_1(x)$ is the matrix

$$P_1 = \left\| \begin{matrix} -P_{1\,p_1-1} & -P_{1\,p_1-2} & -P_{1\,p_1-3} & \cdots & -P_{12} & -P_{11} & -P_{10} \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{matrix} \right\|.$$

So if n is a non-negative integer less than p_1, then the last $p_1 - n$ rows of P_1^{n+1} are the same as the first $p_1 - n$ rows of P_1 itself. Hence, if V_j is the matrix formed from the last r_j rows of $P_j(P_1)$, and if $2 \leqq |\,j\,| \leqq r$, then

$$V_j = \left\| \begin{matrix} P_{jp_j}\cdots P_{j2}P_{j1}P_{j0} & & & \\ & P_{jp_j}\cdots P_{j2}P_{j1}P_{j0} & & \\ & & \cdots\cdots\cdots & \\ & & & P_{jp_j}\cdots P_{j2}P_{j1}P_{j0} \end{matrix} \right\|.$$

This matrix has p_1 columns of which the first p_{-j} are zero, of course.

4. *The Resultant Matrix.* If P_1 is substituted for x in the identity (2), since $P_1(P_1)=0$, it follows that

$$P_k(P_1) = -P_{-k}(P_1)P_0(P_1), \qquad (k = 2, 3, \cdots, r).$$

Now if $W=-P_0(P_1)$, then the last r_k rows of this equation may be written in the form

$$V_k = V_{-k}W, \qquad (k = 2, 3, \cdots, r).$$

Hence if T is defined by the first of the following equations, TW will have the value given by the second of these equations:

$$T = \begin{Vmatrix} V_{-r} \\ \cdots \\ V_{-3} \\ V_{-2} \end{Vmatrix}, \qquad TW = \begin{Vmatrix} V_r \\ \cdots \\ V_3 \\ V_2 \end{Vmatrix}.$$

Let $M_{jk}(-W)$ denote the minor of order j made up from the last j rows, first $j-1$ columns, and (p_1-k)th column of $P_0(P_1)$ for $j=1, 2, \cdots, p_1$, and $k=0, 1, 2, \cdots, p_1-j$. Of course $M_{jk}(-W)=0$ if $j>p_1-p_r$ since the nullity of W is p_r. Now set

$$M_j(-W, x) = \sum_{k=0}^{p_1-j} M_{jk}(-W)x^k, \qquad (j = 1, 2, \cdots, p_1).$$

Because of the triangular form of T, it follows that

$$(3) \qquad M_j(TW, x) = (-1)^j \left[\prod_{k=2}^{s-1} c_{-k}\right](P_{-sp_{-s}})^t M_j(-W, x), \qquad (j = 1, 2, \cdots, p_1 - p_r),$$

where $c_{-k}=(P_{-kp_{-k}})^{r_k}$ and t and s are determined by the inequality $0 \leqq t=j-p_{-s}<r_s$.

An inspection of TW shows that $M_j(TW, x)=0$ unless j is either $p_{-(s+1)}$ or $p_{-s}+1$ for some value of s such that $2 \leqq s \leqq r$. In these exceptional cases, it follows that

$$\left.\begin{aligned} M_{p_{-(s+1)}}(TW, x) &= (-1)^{\sigma_s}\prod_{k=2}^{s} c_k P_s(x)/P_{sp_s} \\ M_{p_{-s}+1}(TW, x) &= (-1)^{\sigma_{s-1}+p_{-s}}\prod_{k=2}^{s} c_k P_s(x)/c_s \end{aligned}\right\}, (s=2, 3, \cdots, r),$$

where σ_k denotes the sum of the products of the integers $r_2, r_3, \cdots, r_k$ taken two at a time, and $\sigma_1=\sigma_2=0$. With the help of (3) these relations yield

$$(4)\quad \left.\begin{aligned} P_s(x)&=(-1)^{\sigma_s+p_{-(s+1)}}P_{sp_s}\Big[\prod_{k=2}^{s} c_{-k}/c_k\Big]M_{p_{-(s+1)}}(-W,x)\\ P_s(x)&=(-1)^{\sigma_{s-1}+1}P_{-sp_{-s}}(c_s/c_{-s})\Big[\prod_{k=2}^{s} c_{-k}/c_k\Big]M_{p_{-s}+1}(-W,x)\end{aligned}\right\},$$

$$(s=2,3,\cdots,r),$$

which shows that the remainders in the euclidean algorithm (1) are *proportional* to the non-zero distinct polynomials in the sequence $M_1(-W, x), M_2(-W, x), \cdots, M_{p_1}(-W, x)$, and the factors of proportionality are independent of x.

Equations (4) may also be written in the equivalent form

$$(5)\quad Q_s(x)=A_sM_{p_{-(s+1)}}(-W,x)=B_sM_{p_{-s}+1}(-W,x),\qquad (s=2,3,\cdots,r),$$

where A_s and B_s are constants independent of x and are given by the relations

$$(6)\quad \left.\begin{aligned} A_s&=(-1)^{\sigma_s+p_{-(s+1)}+s(s-1)/2}P_{sp_s}\prod_{k=2}^{s} c_{-k}/c_k\\ B_s&=(-1)^{\sigma_{s-1}+1+s(s-1)/2}P_{-sp_{-s}}(c_s/c_{-s})\prod_{k=2}^{s} c_{-k}/c_k\end{aligned}\right\},$$

$$(s=2,3,\cdots,r).$$

Expressions equivalent to (4) and (6) would be obtained from them if σ_s were replaced by $q_s=n_s(n_s-1)/2$, where n_s denotes the number of odd integers in the sequence $r_2, r_3, \cdots, r_s$.

5. *The Constants A_s and B_s.* For many applications, it is only necessary to know the remainders in the algorithm to within *positive* factors of proportionality. For example, in order to find the number of real zeros of a polynomial $P_0(x)$ within a given interval, it is sufficient to know the Sturm functions of $P_0(x)$ except for possible *positive* factors. So it is desirable to determine the signs of A_s and B_s in order that the resultant matrix of $P_0(x)$ and $P_0'(x)$ may be used to determine the Sturm functions $P_k(x)$ of $P_0(x)$ for $k=0, 1, \cdots, r$.

If e_k denotes the sign of P_{kp_k}, it follows easily from (1) and the definition of $P_{-k}(x)$ that $e_{k-1}e_{-k}=1$ for $k=2, 3, \cdots, r+1$. For simplicity, let α_s and β_s denote the signs of A_s and B_s respectively. Then

$$(7)\qquad \left.\begin{aligned} \alpha_s &= (-1)^{\sigma_s+p_{-(s+1)}+s(s-1)/2}e_s\prod_{k=2}^{s}(e_ke_{-k})^{r_k} \\ \beta_s &= (-1)^{\sigma_{s-1}+1+s(s-1)/2}e_{-s}\prod_{k=2}^{s-1}(e_ke_{-k})^{r_k} \end{aligned}\right\},\ (s=2,3,\cdots,r).$$

If $r_k\equiv 1 \pmod 2$ for $k=2, 3, \cdots, r$, it follows from (7) that $\alpha_s=\beta_s=1$ for $s=2, 3, \cdots, r$, since in this special case we have $\sigma_s\equiv(s-1)(s-2)/2 \pmod 2$ and $p_{-s}\equiv(s-2) \pmod 2$. In the general case, where not every r_k is odd, it is simpler and more satisfactory to determine β_s than α_s, and so the discussion which follows is given only for β_s.

Let μ_s and ρ_s denote the signs of the leading coefficients of the polynomials $M_{p_{-s}+1}(-W, x)$ and $Q_s(x)$ respectively. It follows that $\rho_s=(-1)^{s(s-1)/2}e_s$ and from (5) that $\rho_s=\beta_s\mu_s$, whence $e_s=(-1)^{s(s-1)/2}\beta_s\mu_s$ for $s=2, 3, \cdots, r$. Now $\beta_2=1, \beta_3=(-\mu_2)^{r_2-1}$, and so the β_{s+1} are given by the recursion formula

$$(8)\qquad \beta_{s+1}=(\beta_s\beta_{s-1}\mu_s\mu_{s-1})^{r_s-1}(-1)^{r_s(p_{-s}+s-1)+1}\beta_s,\qquad (s=3,4,\cdots,r-1).$$

This can be simplified, by treating the odd and even cases separately, to

$$(9)\qquad \left.\begin{aligned} \beta_{s+1} &= (-1)^{m_s}\beta_s \quad \text{if } r_s\equiv 1 \pmod 2 \\ \beta_{s+1} &= -\beta_{s-1}\mu_s\mu_{s-1} \quad \text{if } r_s\equiv 0 \pmod 2 \end{aligned}\right\},\ (s=3,4,\cdots,r-1),$$

where m_s is the number of even integers in the sequence $r_2, r_3, \cdots, r_{s-1}$. Although the argument has been given only for the constants B_s, it is also possible to determine the signs of the constants A_s directly from the resultant matrix in a similar fashion.

6. *Numerical Example.* Consider the two polynomials $Q_0(x)=x^7+2x^5+x^3-x^2-1$ and $Q_1(x)=x^7+x^5-x^2-1$. Then

$$Q_1 = \begin{Vmatrix} 0 & -1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{Vmatrix},$$

$$Q_0(Q_1) = \begin{Vmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{Vmatrix}.$$

It follows immediately that $Q_2(x) = M_1(Q_0(Q_1), x) = x^5 + x^3$. Hence $\mu_2 = 1$, $r_2 = 2$, $p_{-3} = 2$, and $Q_3(x) = (-\mu_2)^{r_3 - 1} M_3(Q_0(Q_1), x) = -(x^2+1)$. Finally, $\mu_3 = 1$, $r_3 = 3$, $p_{-4} = 5$, and so $Q_4(x) = M_6(Q_0(Q_1), x) \equiv 0$. The first remainder is therefore $x^5 + x^3$ and the second and last remainder is $-x^2 - 1$, and this is also the highest common factor of $Q_0(x)$ and $Q_1(x)$.

PRINCETON UNIVERSITY

ON SYMMETRIC DETERMINANTS

BY W. V. PARKER

In a former paper* the writer proved the following theorem:

THEOREM A. *If* $D=|a_{ij}|$ *is a symmetric determinant of order* $n>4$ *with* a_{ij} *real and* $a_{ii}=0$, $(i=1, 2, \cdots, n)$, *and if all fourth-order principal minors of* D *are zero, then* D *vanishes.*

The purpose of this note is to give some results which are obtained immediately from this theorem and which are in one sense a generalization of this theorem.

Suppose D is a symmetric determinant of order $n>4$, with *real* elements, in which all principal minors of order $n-1$ and also all principal minors of order $n-4$ are zero. If $D'=|A_{ij}|$ is the adjoint of D, then $A_{ii}=0$, $(i=1, 2, \cdots, n)$. Each fourth-order principal minor of D' is equal to the product of D^2 by a principal minor of D of order $n-4$.† Therefore D' satisfies the conditions of Theorem A and hence is zero. But $D'=D^{n-1}$ and hence D is also zero and we have the following theorem:

THEOREM 1. *If* D *is a symmetric determinant of order* $n>4$, *with real elements, in which all principal minors of order* $n-1$ *and also all principal minors of order* $n-4$ *are zero, then* D *vanishes.*

Suppose D is a symmetric determinant of order $n>4$, with *real* elements, in which all principal minors of some order $k>3$ and also all principal minors of order $k-3$ are zero. Let M be any $(k+1)$-rowed principal minor of D, $(M=D$ if $n=5)$, then M is a determinant satisfying the conditions of Theorem 1 and hence M is zero. Therefore, in D, all principal minors of order k and also all principal minors of order $k+1$ are zero, hence D is of rank $k-1$ or less.‡ We have thus proved the following theorem:

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* *On real symmetric determinants whose principal diagonal elements are zero*, this Bulletin, vol. 38 (1932), pp. 259–262. See also, *On symmetric determinants*, American Mathematical Monthly, vol. 41 (1934), pp. 174–178.

† Bôcher, *Introduction to Higher Algebra*, p. 31.

‡ Bôcher, loc. cit., page 57, Theorem 2.

THEOREM 2. *If D is a symmetric determinant of order $n>4$, with real elements, in which all principal minors of some order $k>3$ and also all principal minors of order $k-3$ are zero, then D is of rank $k-1$ or less.*

If $n>5$, and $k<n-1$ the minors of Theorem 2 may be divided into two complementary sets such that if all minors of either set are zero the determinant vanishes. This division into sets may be done in n different ways.

Suppose D is a symmetric determinant of order $n>5$, with *real* elements, and M is a principal minor of D of order $n-1$. If all principal minors of M of some order $k>3$ and also all principal minors of M of order $k-3$ are zero, then M is of rank $k-1$ or less by Theorem 2. Let us suppose now that M is in the upper left hand corner of D and expand D according to the products of the elements of the last row and the last column. We get

$$D = a_{nn}M - \sum_{i,j=1}^{n-1} a_{ni}a_{jn}\alpha_{ij},$$

where α_{ij} is the cofactor of a_{ij} in M. If now we make the further restriction that k be less than $n-1$, then, since the rank of M is $k-1$ or less, each $\alpha_{ij}=0$ and consequently $D=0$. We have, therefore, the following result:

THEOREM 3. *If D is a symmetric determinant of order $n>5$, with real elements, and M is a principal minor of D of order $n-1$, and if all principal minors of M of some order k, $3<k<n-1$, and also all principal minors of M of order $k-3$ are zero, then D vanishes.*

Suppose D is a symmetric determinant of order $n>5$, with *real* elements, and that M is a principal minor of D of order $n-1$. Suppose also that all principal minors of D of some order $n-t$ and also all principal minors of D of order $n-t+3$, $(t>3)$, which are not minors of M, are zero. We may assume further, without loss of generality, that M is in the upper left hand corner of D. Let D' be the adjoint of D and M' be the minor of D' corresponding to M in D. Any principal minor of M' of order t (of order $t-3$) is equal to the product of D^{t-1} (D^{t-4}) by the complement in D of the corresponding minor in M. This com-

plementary minor is a minor of D of order $n-t$ $(n-t+3)$ and is not a minor of M and hence is zero by hypothesis. Therefore M' is a symmetric determinant of order $n-1>4$, with *real* elements, in which all principal minors of some order $t>3$ and also all principal minors of order $t-3$ are zero, and hence M' is of rank $t-1$ or less by Theorem 2. If we make the further restriction that t be less than $n-1$ we find, by expanding D' according to the products of the elements of the last row and the last column, that D' is zero. But $D'=D^{n-1}$ and hence D is zero also.

If we write $n-t+3=k$, since $3<t<n-1$, we have $4<k<n$ and hence the truth of the following theorem is apparent:

THEOREM 4. *If D is a symmetric determinant of order $n>5$, with real elements, and M is any principal minor of D of order $n-1$, and if all principal minors of D of some order $k>4$ and also all principal minors of D of order $k-3$, which are not minors of M, are zero, then D vanishes.*

In a second paper the writer* proved a theorem stated as follows:

THEOREM B. *If $D=|a_{ij}|$ is a symmetric determinant of order $n>5$, in which $a_{ii}=0$, $(i=1, 2, \cdots, n)$, and M is any principal minor of D of order $n-1$, then if all fourth order principal minors of D which are not minors of M are zero, D vanishes.*

From this theorem we see that the restriction that the elements of D be real is not necessary in Theorem A when n is greater than five. Consequently the theorems of this paper may be extended to include determinants with complex elements. Theorem 1 is true for complex elements if $n>5$. Theorem 2 is true for complex elements if $n>5$ and $k>4$. Theorems 3 and 4 are true for complex elements if $n>6$ and $4<k<n-1$.

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* *A theorem on symmetric determinants*, this Bulletin, vol. 38 (1932), pp. 545–550.

p-ALGEBRAS OVER A FIELD GENERATED BY ONE INDETERMINATE*

BY A. A. ALBERT

1. *Introduction.* The structure of all division algebras over the simplest type of non-modular field, the field of all rational numbers, has been determined.† The correspondingly simplest type of infinite modular field‡ is the simple transcendental extension $K = F(x)$ of a finite field F. Every division algebra D over such a K is a normal division algebra of degree n over a centrum G which is algebraic of finite degree over K. It is well known that the problem of determining the structure of D is reducible to the case where n is a power of a prime p. When p is the characteristic of F the algebra D is called a p-algebra and we shall solve the problem in this case. Our results will be valid if we replace the finite field F by any perfect field of characteristic p.

The theorem we shall obtain is remarkable not merely because of the character of the result thus derived but also because of the extremely elementary nature of the proof. By using a simple property of the field G described above we shall show that every p-algebra with centrum G is cyclic and of exponent equal to its degree. Moreover this result is due to the unusual fact that all cyclic algebras over G of the same degree p^e have a common pure inseparable splitting field.

2. *Simple Transcendental Extensions of F.* Consider any perfect field F of characteristic p. Then every a of F has the form $a = b^{p^k}$ for b in F. It is easily seen that in fact the correspondences

$$a \longleftrightarrow a^{p^k}, \qquad (k = 0, 1, \cdots),$$

are automorphisms of F.

We let x be an indeterminate over F, $J = F[x]$ be the set of

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* Presented to the Society, September 7, 1937.

† Cf. the paper of H. Hasse and the author, Transactions of this Society, vol. 34 (1932), pp. 722–726.

‡ There is no structure problem for division algebras over finite fields as they are always finite fields.

all polynomials in x with coefficients in F. Then the rational function field $K=F(x)$ of all rational functions of x with coefficients in F is the quotient field of J. If α is in J such that

$$\alpha = a_0 + a_1 x + \cdots + a_n x^n, \qquad (a_i \text{ in } F),$$

then $a_i = b_i^q$ with $q=p^k$ and b_i in F, and

$$\alpha = \beta^q, \qquad \beta = b_0 + b_1 y + \cdots + b_n y^n$$

is in $F[y]$, $y^q=x$. Evidently every quantity of the field $K=F(x)$ is the p^k-th power of a quantity of $K_0=F(y)$. This result will be shown to imply the following theorem:

THEOREM 1. *Let x be an indeterminate over a perfect field F of characteristic p so that every algebraic extension G of degree n over $K=F(x)$ is inseparable of degree $t=p^e$ over its maximal separable sub-field $H=K(u)$ of degree $m=np^{-e}$ over K. Then*

$$(1) \qquad G = H(y) = K_0(u) = F(y, u), \quad K_0 = F(y), \quad y^t = x,$$

so that G is uniquely determined in the sense of equivalence by H and $t=p^e$. Conversely the field G of (1) *has degree p^e over H.*

If $q=p^k$ and $K(u^q)$ were a proper sub-field of $H=K(u)$, the field H would be inseparable. Hence every quantity of H of degree m over K is *uniquely* expressible in the form

$$(2) \qquad w = \alpha_0 + \alpha_1 u^q + \cdots + \alpha_{m-1} u^{q(m-1)}, \qquad (\alpha_i \text{ in } K).$$

If $K(u, x^{1/p})$ did not have degree p over $K(u)$, the quantity x would have the form w^p with w in $K(u)$. Apply (2) with $k=0$ and obtain $x=w^p=\alpha_0^p+\alpha_1^p u^p+\cdots+\alpha_{m-1}^p u^{p(m-1)}$. The uniqueness of the expression (2) for $k=1$ implies that $\alpha_1=\cdots=\alpha_{m-1}=0$, $x=\alpha_0^p$ is the quotient of two polynomials in x^p, which is clearly impossible. Hence $H(x^{1/p})$ has degree p over H, and an evident induction implies that if $y^{p^e}=x$ then $H(y)=F(y, u)$ has degree p^e over H.

We now let G_1 be algebraic of finite degree over K so that G_1 has a maximal separable sub-field H and degree $t=p^e$ over H. It is conceivably not a simple extension of H. Without loss of generality we assume that G_1 is contained in a field which also contains the quantity y and hence the field $H(y)=G$. Every z of G_1 has the property $z^t=w$ in H, and if w has the form (2)

for $q=t$ then each $\alpha_i=\beta_i{}^t$ with β_i in $K_0=F(y)$, $w=v^t$, where $v=\beta_0+\beta_1 u+\cdots+\beta_{m-1}u^{m-1}$ is in $K_0(u)=G$. Thus $z^t=v^t$, $z=v$ is in G, $G_1 \leqq G$. Since G and G_1 have the same degree over H, we have $G_1=G$.

An immediate corollary of our proof may be stated as follows:*

THEOREM 2. *Let H be separable of finite degree over $K=F(x)$ and g in H, $y^t=x$ where $t=p^e$. Then $g=d^t$ for d in $F(y)=K(y)$ of degree t over K.*

3. *The Determination.* The fundamental result on p-algebras over fields of our simple type may be thus stated:

THEOREM 3. *Let D be a normal division algebra of degree p^e over a field G which is algebraic of finite† degree over a simple transcendental extension of a perfect field F of characteristic p. Then G is separable of finite degree over $F(x)$, x an indeterminate over F, and D is a cyclic algebra*

$$(Z, S, x) \text{ over } G. \tag{3}$$

The exponent‡ of D is its degree p^e so that conversely the cyclic algebra (3) is a division algebra if and only if x is not the norm of any quantity of the cyclic sub-field of Z of degree p over G.

The now standard notation (3) in our case means that Z is a cyclic field of degree $p^e=t$ over G with generating automorphism

$$S: \qquad z \longleftrightarrow z^S, \qquad (z, z^S \text{ in } Z),$$

and that every quantity of D has the form

$$z_0+z_1y+\cdots+z_{t-1}y^{t-1}, \qquad (z_i \text{ in } Z).$$

* F. K. Schmidt, *Analytische Zahlentheorie in Körpern der Characteristik p*, Mathematische Zeitschrift, vol. 33 (1931), pp. 1–32, p. 8, showed that every algebraic field of finite degree over a simple transcendental extension of a finite field F is separable over $F(x)$, for some indeterminate x. This inexplicit form of our Theorems 1, 2 is of course insufficient for our proof of Theorem 3.

† The results of Theorem 3 are also true for the case when G is algebraic but not of finite degree over K. For the method which leads to the reduction to the case of finite degree see the author's paper in this Bulletin, vol. 39 (1933), pp. 746–749.

‡ The result that the exponent and degree are equal was proved for *arbitrary degree n* by E. Witt, Mathematische Annalen, vol. 110 (1934), pp. 12–28. However his proof requires the same formidable machinery used for the analogous result on algebras over algebraic number fields, whereas the proof we give here is almost trivial.

Moreover $y^r z = z^{S^r} y^r$ for every z of Z, $y^t = x$. Thus (3) states that all cyclic algebras of degree $t = p^e$ over G have $G(y)$ as a common splitting field (equivalent to a maximal sub-field of all such algebras).

For proof we notice that the result concerning G is clearly part of Theorem 1. Let $A = (Y, T, g)$ be any cyclic algebra of degree $s = p^f \leqq p^e$ over G. By Theorem 2 we have $g = d^s$ with d in $L = G(x^{1/s})$, $A_L = (Y_L, T, 1)$ is a total matric algebra over L. Thus L is a splitting field* of every cyclic algebra of degree p^f. It is known* that D has exponent $p^f \leqq p^e$ over G and also that† then D is similar to a direct product of cyclic algebras A_i of degree p^f. But L splits each A_i and hence the algebra D. Since* the degree of every splitting field of D is divisible by the degree p^e of D it follows that $p^e \leqq p^f$, $t = p^e = p^f$. Thus D has exponent p^e and $L = G(x^{1/t})$ as a splitting field. But then it is known‡ that D has the form (3). The final statement in the theorem is a known§ consequence of the result that D of (3) is a division algebra if and only if its exponent is its degree.

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* These results are due to R. Brauer, E. Noether, and the author and are now well known. Their proofs with references may be found in the Ergebnisse tract of M. Deuring on *Algebren*.

† This is Theorem 18 of the author's paper in the Transactions of this Society, vol. 40 (1936), pp. 112–126.

‡ Loc. cit. See the proof of Theorem 12.

§ Cf. American Journal of Mathematics, vol. 54 (1932), pp. 1–13 for proof.

THE FOUR-VERTEX THEOREM FOR A CERTAIN TYPE OF SPACE CURVES.*

BY W. C. GRAUSTEIN AND S. B. JACKSON

It is the purpose of this note to establish for a certain type, A, of space curves the new form of the four-vertex theorem recently stated and proved for plane ovals.†

A space curve C shall be said to be of type A provided (a) it is a closed regular curve of class C'', (b) its curvature never vanishes, (c) the projection of its tangent indicatrix I on a plane π perpendicular to the line joining the origin O to the center of gravity G of I is an oval or, if G coincides with O, the projection of I on some plane, π, is an oval, and (d) this oval is traced just once when C is traced once.

By a vertex shall be meant a point, or an arc of constant curvature, for which the curvature has a relative extremum with respect to the neighboring arcs on either side. A vertex shall be said to be *primary* if the curvature at it has a maximum (minimum) which is greater (less) than the average curvature of the curve with respect to the arc. Otherwise, a vertex shall be termed *secondary*.‡

The theorem to be established may now be formulated as follows.

STATEMENT I. *On a curve of type A, whose curvature is not constant, there are at least four primary vertices. More precisely, the number of primary vertices, if finite, exceeds the number of secondary vertices by at least four, and is infinite if the number of secondary vertices is infinite.*

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* Presented to the Society, March 26, 1937.

† W. C. Graustein, *A new form of the four-vertex theorem*, Monatshefte für Mathematik und Physik, vol. 43 (1936), pp. 381–384; for a related theorem, see Hayashi, *Some general applications of Fourier series*, Rendiconti del Circolo Matematico di Palermo, vol. 50 (1926), p. 100. For other work on the four-vertex theorem in space see Süss, *Ein Vierscheitelsatz bei geschlossenen Raumkurven*, Tôhoku Mathematical Journal, vol. 29 (1928), pp. 359–362; Takasu, *Vierscheitelsatz für Raumkurven*, Tôhoku Mathematical Journal, vol. 39 (1934), pp. 292–298, and vol. 41 (1936), pp. 317, 318.

‡ Ibid., p. 381.

The theorem may be stated more simply in terms of the concept of a transition of the curvature.* If $1/R$ is the curvature of the curve and $1/A$ the average curvature with respect to the arc, a transition of the curvature is defined as a point, or arc, of the curve for which $1/R$ has the value $1/A$ and $1/R-1/A$ changes sign, that is, has opposite signs for neighboring arcs on either side.

A transition of the curvature is never a vertex, by definition. Moreover, from the continuity of $1/R-1/A$ it follows that the number of primary vertices between two consecutive transitions of the curvature exceeds the number of secondary vertices on this arc by exactly one, or both types of vertices are infinite in number.† Hence our theorem may be restated as follows:

STATEMENT II. *On a curve of type A, whose curvature is not constant, there are at least four transitions of the curvature.*

In the proof of the theorem we shall prefer to deal with the radius of curvature of the given curve C rather than with the curvature and to employ, as parameter, not the arc s of C, but the arc ϕ of the tangent indicatrix I. If ϕ is measured in the direction of increasing s, we have

$$\frac{1}{R} = \frac{d\phi}{ds}.$$

Hence, since $1/R>0$, ϕ is an admissible parameter for both C and I. Furthermore, since $1/R$ is a continuous function of s which is periodic of period D—the length of C, R is a single-valued continuous function of ϕ which is periodic of period d—the length of I.

From the relations $ds=Rd\phi$ and $d\phi=ds/R$, it follows that

$$\int_0^d Rd\phi = D, \qquad \int_0^D \frac{ds}{R} = d.$$

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* Ibid., p. 382.

† This statement may be expressed quantitatively by the equation

$$P=S+T$$

where P, S, and T denote, respectively, the number of primary vertices, secondary vertices, and transitions of curvature, and where, furthermore, it is understood that if S or T is infinite the equation is simply to mean that P is infinite.

Consequently, the average radius of curvature of C with respect to the arc ϕ of I, namely $A = D/d$, is the reciprocal of the average curvature of C with respect to the arc s of C, namely, $1/A = d/D$. Thus the definitions of primary and secondary vertices and that of a transition of the curvature might just as well have been based on comparisons of the radius of curvature with the average radius of curvature with respect to ϕ.

The tangent indicatrix, I, of the curve C has the parametric equations

$$y_i = \alpha_i(\phi), \qquad (i = 1, 2, 3),$$

where $\alpha_1, \alpha_2, \alpha_3$ are the direction cosines of the directed tangent to C at the point $\phi = \phi$. For C itself we have the equations

$$dx_i = R\alpha_i d\phi, \qquad (i = 1, 2, 3),$$

with the conditions of closure

$$\int_0^d R\alpha_i d\phi = 0, \qquad (i = 1, 2, 3).$$

These conditions, together with the fact that A is the average value of R with respect to ϕ, may be written in the forms

$$\int_0^d (R - A)d\phi = 0, \quad \int_0^d (R - A)\alpha_i d\phi = -D\bar{\alpha}_i, \quad (i = 1, 2, 3),$$

where $(\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3)$ are the coördinates of the center of gravity, G, of the curve I, and hence in the form of the identity

$$\int_0^d (a_0 + a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3)(R - A)d\phi \equiv -D(a_1\bar{\alpha}_1 + a_2\bar{\alpha}_2 + a_3\bar{\alpha}_3)$$

in the arbitrary constants a_0, a_1, a_2, a_3.

The right-hand side of this identity vanishes if the direction with the components a_1, a_2, a_3 is perpendicular to the line OG or if, as in the case of a plane oval, G coincides with O. Thus, we pass to the following conclusion.

LEMMA 1. *The relation*

$$\int_0^d (a_0 + a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3)(R - A)d\phi = 0$$

holds for every plane $a_0+a_1x_1+a_2x_2+a_3x_3=0$ *which is perpendicular to the plane* π *associated with the given curve C of type A after the manner described in the definition of a curve of this type.*

We may now establish our theorem, by assuming that it is false. There are, then, since $R \not\equiv A$, just two transitions of the curvature. In other words, there exist two points P, Q on I such that $R-A$ is of one sign or zero for one of the arcs, PQ, into which the points divide I and is of the opposite sign or zero for the other arc, QP.

Consider, next, the function $a_0+a_1\alpha_1+a_2\alpha_2+a_3\alpha_3$, where the a's are so chosen that $a_0+a_1x_1+a_2x_2+a_3x_3=0$ represents the plane through P and Q which is perpendicular to π. This plane cuts the oval which is the projection of I on π in just two points, which are necessarily the projections of P and Q, and of P and Q only. Hence, it divides the oval into two arcs, one on either side of it, and these arcs must be the projections of the arcs PQ and QP of I. It follows, then, that the function in question is positive or zero for one of the latter arcs, and negative or zero for the other.

It is now evident that the expression

$$(a_0 + a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3)(R - A),$$

never changes sign, and so the integral of it extended around I cannot be zero. But this contradicts Lemma 1 and our theorem is established.

The reader may observe that, when the center of gravity of I is at the origin, it is unnecessary to assume that there exists a plane on which the projection of I is an oval. It is sufficient to make the less restrictive assumption that through each two points of I there passes a plane, p, dividing I into exactly two arcs, one on either side of p. This follows from the fact that, when $\bar{a}_i=0$, $i=1, 2, 3$, Lemma 1 holds for any plane whatever.

The existence of twisted curves of type A is readily established. Fujiwara has shown that a closed curve on the unit sphere is the tangent indicatrix of a closed space curve if and only if the center, O, of the sphere lies interior to the convex hull of the curve.* The problem of exhibiting a twisted space

* M. Fujiwara, *Über die kleinsten eine Kurve enthaltenden konvexen Körper*, The Science Reports of the Tôhoku Imperial University, vol. 4 (1915), p. 341.

curve of type A is thus reduced to that of finding a spherical curve, other than a great circle, whose convex hull has O as an interior point and which projects into an oval on a properly chosen plane.

On the unit sphere $x_1=\sin\phi\cos\theta$, $x_2=\sin\phi\sin\theta$, $x_3=\cos\phi$, consider the curve I_δ:

$$\phi = \frac{\pi}{2} + \delta(\tfrac{1}{2} + \sin 2n\theta),$$

where n is a positive integer and δ is a small positive constant. This curve cuts through the (x_1, x_2)-plane at $4n$ points regularly distributed in pairs about the circle $\phi=\pi/2$, and hence O is interior to the convex hull of these points. A similar statement can then be made concerning the point $(0, 0, \epsilon)$ with respect to the points in which the plane $x_3=\epsilon$ meets I_δ, provided $\epsilon \lesseqgtr 0$ is sufficiently small in numerical value. Consequently, O is interior to the convex hull of I_δ.

Since I_δ is symmetric in the x_3-axis, its center of gravity lies on this axis. That G does not coincide with O is evident from the fact that the maximum value of ϕ is $\pi/2+(3/2)\delta$, whereas the minimum value is $\pi/2-\delta/2$. Hence, a plane π of the definition of a curve of type A is in this case a horizontal plane, say, the (x_1, x_2)-plane. But, for δ sufficiently small, the projection of I_δ on the (x_1, x_2)-plane is certainly an oval, inasmuch as the curvature of the projection is a continuous function of δ and, when δ approaches zero, the projection approaches the circle $\phi=\pi/2$. Therefore, the curve I_δ, for sufficiently small values of δ, is the tangent indicatrix of a twisted curve of type A.

An example of a curve on the unit sphere which is the tangent indicatrix of a twisted curve of type A and has its center of gravity at O is given by $\phi=\pi/2+\delta\sin 2n\theta$, where n is a positive integer and δ is a positive constant, sufficiently small.

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See also Fenchel, *Geschlossene Raumkurven mit vorgeschriebenen Tangentenbild*, Jahresbericht der Deutschen Mathematiker Vereinigung, vol. 39 (1930), pp. 183-185.

PROOF OF THE NON-ISOMORPHISM OF TWO COLLINEATION GROUPS OF ORDER 5184*

BY F. A. LEWIS

Introduction. Let S denote the collineation

$$\rho x_r = \epsilon^{r-1} x_r', \quad (r = 1, \cdots, n), \qquad \epsilon = \cos(2\pi/n) + i \sin(2\pi/n),$$

and T the collineation

$$\rho x_r = x_{r+1}', \qquad (r = 1, \cdots, n), \qquad x_{n+1}' \equiv x_1'.$$

The abelian group $\{S, T\}$ of order n^2 is invariant under a group† C_n of order

$$n^5\left(1 - \frac{1}{p_1^2}\right)\left(1 - \frac{1}{p_2^2}\right) \cdots \left(1 - \frac{1}{p_m^2}\right),$$

where $p_1, p_2, \cdots, p_m$ are the distinct prime factors of n. The order of C_6 is 5184.

Winger‡ has discussed briefly the monomial group of order $(r+1)!n^r$ that leaves invariant the variety

$$x_0^n + x_1^n + x_2^n + \cdots + x_r^n = 0.$$

This group is generated by the symmetric group of degree $r+1$ and an abelian group of order n^r in canonical form. For $r=3$ and $n=6$ there results a group G of order 5184 which has been treated by Musselman.§ The purpose of this note is to prove that G and C_6 are not simply isomorphic. The proof consists in showing that the number of collineations of period 2 in G exceeds the number of collineations of period 2 in C_6.

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* Presented to the Society, June 18, 1936.

† In fact, C_n is the largest collineation group in n variables containing $\{S, T\}$ invariantly, the coefficients and variables being in the field of complex numbers. (Author's dissertation, Ohio State University, 1934.)

‡ *Trinomial curves and monomial groups*, American Journal of Mathematics, vol. 52 (1930), p. 394.

§ *On an imprimitive group of order* 5184, American Journal of Mathematics, vol. 49 (1927).

Proof of the Non-Isomorphism of G and C_6. The group C_6 is generated by $\{S, T\}$ and the two collineations

$$V: \quad \rho x_r = \sum_{c=1}^{6} \epsilon^{(r-1)(c-1)} x_c', \quad (r = 1, \cdots, 6),$$

$$W: \quad \rho x_r = \epsilon^{-(r-1)^2/2} x_r', \qquad (r = 1, \cdots, 6),$$

satisfying the following relations:

$$V^4 = W^{12} = 1, \ V^2W = WV^2, \ V^{-1}SV = T^{-1}, \ W^{-1}SW = S,$$

$$(VW)^3 = V^2 = (WV)^3, \ W^6 = S^3, \ V^{-1}TV = S, \ W^{-1}TW = S^{-1}T.$$

The order of $H = \{V, W\}$ is 576. This group may be constructed by the following chain of invariant subgroups and an independent proof that the order of C_6 is 5184 follows readily.

$$H = \{V, G_{288}\}, \quad G_{288} = \{W^5VW^3V^3, G_{96}\}, \quad G_{96} = \{W^2, G_{32}\},$$

$$G_{32} = \{W^2(W^2V)^2, G_{16}\}, \ G_{16} = \{(W^2V)^3V, G_4\}, \ G_4 = \{S^3, T^3\}.$$

Since G_4 is contained in $\{S, T\}$ which is invariant under H, the order of C_6 is $576 \cdot 36/4 = 5184$.

If Q, of order 144, represents the quotient group of C_6 with respect to $\{S, T\}$, each element of Q, being a co-set of C_6, represents 36 collineations of C_6 that transform $\{S, T\}$ into itself according to the same isomorphism of $\{S, T\}$ with itself.* There are 24 collineations S^iT^k of period 6 in $\{S, T\}$; if S is transformed into a particular S^iT^k, the collineation S^lT^m into which T is to be transformed may be selected in six ways. Let K represent a class of 144 collineations of C_6 corresponding to the 144 distinct possible sets (j, k, l, m). That is, K contains one and only one collineation from each of the 144 augmented co-sets of C_6. The square of $A \cdot S^rT^s$, an arbitrary collineation of the class K from the co-set to which A belongs, may be expressed in the form $A^2S^uT^v$ and hence is of period 2 only if A^2 is in $\{S, T\}$. That is, a necessary condition that $A \cdot S^rT^s$ be of period 2 is that A^2 be commutative with both S and T. Among any class K there are only 8 collineations B such that the corresponding sets of values (j, k, l, m) satisfy the congruences arising from the conditions that B^2 transform S into S and T into T.

* It may easily be proved that the 36 collineations of $\{S, T\}$ are the only collineations in six variables commutative with both S and T.

The following table shows 8 such collineations, their squares, and the collineations of $\{S, T\}$ which multiply these 8 collineations on the right to form collineations of C_6 of period 2. The numbers in the last column show the total number of collineations of C_6 of period 2 corresponding to each B of K. Thus it is seen that C_6 contains just 99 collineations of period 2.

It is easily shown that G contains more than 99 collineations of period 2 and hence G and C_6 are not simply isomorphic.

			W^2				$W^6=S^3$	$T^3,\ S^3T^3$	2
			S^3				$S^6=1$	$1,\ T^3,\ S^3T^3$	3
			$U^3=V^{-1}W^2V$				$U^6=T^3$	$S^3,\ S^3T^3$	2
$X=WVWV^2WV^2$	-1	0	0	1	0	0			
	0	0	1	0	0	1		S^jT^k where (j, k) satisfies the congruence	
	0	1	0	0	-1	0			
	1	0	0	1	0	0	$X^2=1$	$3(j+k)\equiv 0 \pmod 6$	18
	0	0	-1	0	0	1			
	0	1	0	0	1	0			
			$V^2=R$				$V^4=1$	$S^jT^k,\ (j, k=1, \cdots, 6)$	36
			$RX=XR$				$(RX)^2=1$	$1,\ S^3T^3$ $(j=1, \cdots, 6)$	2
			$RW^2=W^2R$				$(RW^2)^2=W^6$	S^jT^k $(k=1, 3, 5)$ $(j=1, 3, 5)$	18
			$RU^2=U^2R$				$(RU^2)^2=T^3$	S^jT^k $(k=1, \cdots, 6)$	18
									99

UNIVERSITY OF ALABAMA

OFFICIAL COMMUNICATIONS

Meetings of the Society have been fixed at the following times and places:

NEW YORK CITY, October 30, 1937.

Abstracts must be in the hands of Associate Secretary T. R. Hollcroft, 531 West 116th St., New York, N. Y., not later than October 2. Abstracts received by the Secretaries not later than September 3 have appeared in the September issue of this Bulletin. By invitation of the Program Committee Professor D. J. Struik will deliver an address entitled *The application of tensor analysis to problems of electrical engineering.*

IOWA CITY, IOWA, November 26–27, 1937.

Abstracts must be in the hands of Associate Secretary M. H. Ingraham, University of Wisconsin, Madison, Wisconsin, not later than October 30; they will appear in the November issue of this Bulletin. By invitation of the program committee, addresses will be delivered by Professor W. J. Trjitzinsky on *Singular point problems in the theory of linear differential equations*, and by Professor A. H. Copeland on *Fundamental concepts of the theory of probability*.

PASADENA, CALIFORNIA, November 27, 1937.

Abstracts must be in the hands of Associate Secretary T. M. Putnam, University of California, Berkeley, California, not later than October 30; they will appear in the November issue of this Bulletin.

INDIANAPOLIS, INDIANA, ANNUAL MEETING, December 28–30, 1937.

Abstracts must be in the hands of Associate Secretary T. R. Hollcroft, 531 West 116th St., New York, N. Y., not later than November 23, 1937. Abstracts received by the Secretaries by November 2 will appear in the November issue of this Bulletin. The annual Gibbs Lecture will be delivered by Professor Charles A. Kraus, on *The present status of the theory of electrolytes*, before a joint session of this Society and the American Association for the Advancement of Science. Professor G. C. Evans will deliver his retiring address as Chairman of Section A of the latter organization at a joint meeting of Section A with this Society and the Mathematical Association of America. His subject will be *Mathematical progress in theoretical economics*. There will also be a joint session of this Society with the Association for Symbolic Logic, and one with the Institute of Mathematical Statistics. By invitation of the Program Committee, Professor Alonzo Church will deliver an address entitled *The constructive second number class.*

R. G. D. RICHARDSON, *Secretary of the Society.*

Articles for insertion in the BULLETIN should be addressed to the Managing Editor, D. R. CURTISS, Northwestern University, Evanston, Ill. Reviews should be sent to W. R. LONGLEY, Yale University, New Haven, Conn. Notes should be sent to H. W. KUHN, Ohio State University, Columbus, Ohio.

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